

# Invariance Matrices and Twelve-Tone Rows

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## The ordered intervals between two twelve-tone rows

Let P and Q be two twelve-tone rows. There are 144 distinct pairs of pcs between P and Q, one pc in P and the other pc in Q. Let p be a pc in P and q be a pc in Q. The ordered interval from p to q is  $q-p$ .<sup>1</sup> This is distinct from the interval from q to p, which is  $p-q$ ; and if the interval from p to q is n, the interval from q to p is  $-n$ .

We will now use subscripts to denote the pcs in P and Q.  $P_0$  is the first pc on P,  $P_1$  is the second pc of P,  $P_2$  is the third pc of P,  $P_B$  is the twelfth pc of P. Likewise for Q. (So  $Q_6$  is the seventh pc of Q.) So, the interval from  $P_m$  to  $Q_n$  is  $Q_n - P_m$ .

We use a matrix  $E(P,Q)$  to list all 144 intervals from P to Q.<sup>2</sup> We label the cells of the matrix in the same way we label the pcs of P and Q. So  $E_{0,0}$  is the cell in the first row and the first column of  $E(P,Q)$ .  $E_{3,4}$  is the cell in the 4<sup>th</sup> row and the 5<sup>th</sup> column of  $E(P,Q)$ . When we want to indicate what interval is in a cell of the matrix, we write  $h = E_{x,y}$ , which means that the interval h is in the  $x$ th+1 row and  $y$ th+1 column of the matrix  $E$ .<sup>3</sup>

Examples: Using the matrix on the next page, we can indicate that 8 is in the first row and column of the matrix by writing  $8 = E_{0,0}$ . 4 is in the top row and 6<sup>th</sup> column of the matrix; this is written  $4 = E_{0,5}$ . What is in  $E_{9,7}$ ? It is 0, in the 10th row and 8<sup>th</sup> column of the matrix. The positions  $E_{x,x}$  are the set of cells (for all x) that lie on the main diagonal of E, from the upper left corner cell to the lower right corner cell, in other words, from  $E_{0,0}$  to  $E_{B,B}$ .

The matrix  $E(P,Q)$  lists the interval from  $P_m$  to  $Q_n$  in the cell  $E_{m,n}$ . We write the row P to the left of the matrix rows and Q above the matrix columns, as shown on the next page.

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<sup>1</sup> In this section, all addition and subtraction is done mod-12.

<sup>2</sup> Later, we will show that the  $E(P,Q)$  matrix is the same as the T-matrix between P and Q.

<sup>3</sup> When the context is clear, we simply write E for  $E(P,Q)$  for easier reading.

Let <038749BA2561> be row P and <8730B495261A > be row Q. E(P,Q) is shown below.

	8	7	3	0	B	4	9	5	2	6	1	A
0	8	7	3	0	B	4	9	5	2	6	1	A
3	5	4	0	9	8	1	6	2	B	3	A	7
8	0	B	7	4	3	8	1	9	6	A	5	2
7	1	0	8	5	4	9	2	A	7	B	6	3
4	4	3	B	8	7	0	5	1	A	2	9	6
9	B	A	6	3	2	7	0	8	5	9	4	1
B	9	8	4	1	0	5	A	6	3	7	2	B
A	A	9	5	2	1	6	B	7	4	8	3	0
2	6	5	1	A	9	2	7	3	0	4	B	8
5	3	2	A	7	6	B	4	0	9	1	8	5
6	2	1	9	6	5	A	3	B	8	0	7	4
1	7	6	2	B	A	3	8	4	1	5	0	9

For instance, the matrix shows the interval 8 from  $P_3$  to  $Q_2$  in cell  $E_{3,2}$ ;  $E_{3,2} = 8 = Q_2 - P_3 = 3 - 7$ .

Now let us align P and Q pc-to-pc with intervals from corresponding positions in P to Q underneath as shown here.

038749BA2561 = row P  
 8730B495261A = row Q  
 847577A70179 = intervals from  $P_n$  to  $Q_n$  for all n.

The bottom line of intervals can be found on the main diagonal of E (since  $E_{n,n}$  = intervals from  $P_n$  to  $Q_n$ ). This diagonal is shown in bold face in the following copy of the matrix.

	8	7	3	0	B	4	9	5	2	6	1	A
0	<b>8</b>	7	3	0	<u>B</u>	4	9	5	2	6	1	<u>A</u>
3	5	<b>4</b>	0	9	8	<u>1</u>	6	2	B	3	<u>A</u>	7
8	0	B	<b>7</b>	4	3	8	<u>1</u>	9	6	<u>A</u>	5	2
7	1	0	8	<b>5</b>	4	9	<u>2</u>	<u>A</u>	7	B	6	3
4	4	3	B	8	<b>7</b>	0	5	<u>1</u>	<u>A</u>	2	9	6
9	B	A	6	3	2	<b>7</b>	0	8	5	<u>9</u>	4	1
B	9	8	4	1	0	5	<b>A</b>	6	3	7	<u>2</u>	B
A	A	9	5	2	<u>1</u>	6	B	<b>7</b>	4	8	3	<u>0</u>
2	6	5	1	<u>A</u>	9	2	7	3	<b>0</b>	4	B	<u>8</u>
5	3	2	<u>A</u>	7	6	B	4	0	9	<b>1</b>	8	5
6	2	<u>1</u>	9	6	5	A	3	B	8	0	<b>7</b>	4
1	7	6	2	B	A	3	8	4	1	5	0	<b>9</b>

Now we realign P with Q shifted one position to the left.

038749BA2561 = row P  
8730B495261A = row Q  
70440064484 = intervals from row P to row Q shifted 1 position to the left.

The bottom line of intervals can be found on the diagonal to the immediate right of the main diagonal of E (since  $E_{n,n+1}$  = intervals from  $P_n$  to  $Q_{n+1}$ ). This is the diagonal to the right of the bold faced diagonal in the matrix just given.

Now we align P with Q shifted one position to the right and take the intervals likewise.

038749BA2561 = row P  
8730B495261A = row Q  
5B8825B3900 = intervals from row P to row Q shifted -1 position to the left.

The bottom line of intervals can be found on the diagonal to the immediate left of the main diagonal of E (since  $E_{n,n-1}$  = intervals from  $P_n$  to  $Q_{n-1}$ ). This is the diagonal to the left of the bold faced diagonal in the matrix given above.

In general,

*the xth diagonal to the right of the main diagonal of E contains the intervals from row P to row Q shifted x positions to the left.*

One more example illustrates this. Align P and Q shifted x positions to the left, where  $x = 4$ . This is read off E from the diagonal 4 to the right of the main diagonal. This is the diagonal that is underlined in the matrix given above

038749BA2561 = row P  
8730B495261A = row Q  
B11AA920 intervals from row P to row Q shifted 4 positions to the left.

If we align row P with RQ, we have a new series of intervals.

038749BA2561 = row P  
A162594B0378 = row RQ  
AAA71051AA17 intervals from row P to row RQ.

This series of intervals are found on the secondary diagonal of the matrix, reading down from the upper right hand corner to the lower left hand corner of the matrix, shown in italics on the matrix given above.

We can also shift RQ to the left or right and find a diagonal of the matrix that registers the intervals between row P and the shifted RQ row:

*the xth diagonal to the right of the secondary diagonal of E contains the intervals from row R to row RQ shifted x positions to the right.*

### E(P,Q) and Hadamard matrices

Let us now compare the Hadamard matrix for P and Q, H(P,Q), and the E(P,Q) matrix for the same rows.

This is the Hadamard matrix H(P,Q):

	8	7	3	0	B	4	9	5	2	6	1	A
0				1								
3			1									
8	1											
7		1										
4						1						
9							1					
B					1							
A												1
2									1			
5								1				
6										1		
1											1	

This is the E(P,Q) matrix with the 0 entries in bold face:

	8	7	3	0	B	4	9	5	2	6	1	A
0	8	7	3	<b>0</b>	B	4	9	5	2	6	1	A
3	5	4	<b>0</b>	9	8	1	6	2	B	3	A	7
8	<b>0</b>	B	7	4	3	8	1	9	6	A	5	2
7	1	<b>0</b>	8	5	4	9	2	A	7	B	6	3
4	4	3	B	8	7	<b>0</b>	5	1	A	2	9	6
9	B	A	6	3	2	7	<b>0</b>	8	5	9	4	1
B	9	8	4	1	<b>0</b>	5	A	6	3	7	2	B
A	A	9	5	2	1	6	B	7	4	8	3	<b>0</b>
2	6	5	1	A	9	2	7	3	<b>0</b>	4	B	8
5	3	2	A	7	6	B	4	<b>0</b>	9	1	8	5
6	2	1	9	6	5	A	3	B	8	<b>0</b>	7	4
1	7	6	2	B	A	3	8	4	1	5	<b>0</b>	9

We see that the pattern of zeros in E(P,Q) is the same as the 1s in the H(P,Q). A zero interval between two pcs indicates that the pcs are the same; thus the pattern of intersection shown by 1s on the Hadamard matrix is same as the pattern of zero intervals on the E(P,Q) matrix.

Now let us compare a new Hadamard matrix made from  $T_1P$  and  $Q$  with the  $E(P,Q)$  matrix with the 1s in boldface.

	8	7	3	0	B	4	9	5	2	6	1	A
1											<b>1</b>	
4						<b>1</b>						
9							<b>1</b>					
8	<b>1</b>											
5								<b>1</b>				
A												<b>1</b>
0				<b>1</b>								
B					<b>1</b>							
3			<b>1</b>									
6										<b>1</b>		
7		<b>1</b>										
2											<b>1</b>	

	8	7	3	0	B	4	9	5	2	6	1	A
0	8	7	3	0	B	4	9	5	2	6	<b>1</b>	A
3	5	4	0	9	8	<b>1</b>	6	2	B	3	A	7
8	0	B	7	4	3	8	<b>1</b>	9	6	A	5	2
7	<b>1</b>	0	8	5	4	9	2	A	7	B	6	3
4	4	3	B	8	7	0	5	<b>1</b>	A	2	9	6
9	B	A	6	3	2	7	0	8	5	9	4	<b>1</b>
B	9	8	4	<b>1</b>	0	5	A	6	3	7	2	B
A	A	9	5	2	<b>1</b>	6	B	7	4	8	3	0
2	6	5	<b>1</b>	A	9	2	7	3	0	4	B	8
5	3	2	A	7	6	B	4	0	9	<b>1</b>	8	5
6	2	<b>1</b>	9	6	5	A	3	B	8	0	7	4
1	7	6	2	B	A	3	8	4	<b>1</b>	5	0	9

This time the correspondence between the two matrices is between the pattern of 1s in  $H(T_1P,Q)$  and the 1s in the  $E(P,Q)$  matrix.

This suggests a theorem:

*The pattern of ns in  $E(P,Q)$  is the same as the pattern of 1s in  $H(T_nP,Q)$ .*

Thus the  $E(P,Q)$  contains 12 Hadamard matrices, between the transpositionally related members of the row-class of  $P$  and the row  $Q$ .

We might want to know the permutational relations between the rest of the other members of the row-class of  $P$  and the row  $Q$ . The Hadamard matrices  $H(RT_nP,Q)$  are determined by reading the  $E(P,Q)$  upside down. This leaves the  $T_nIP$  and  $RT_nIP$  members of the row-class of  $P$ . We simply make a new matrix  $E(IP,Q)$ . The Hadamard matrices of

the 12  $T_n$ IP forms and Q are located in the patterns of ns in this new matrix and we read these patterns upside down to get the Hadamard matrices of the  $RT_n$ IP forms and Q.<sup>4</sup>

Below is the E(IP,Q) matrix.

	8	7	3	0	B	4	9	5	2	6	1	A
0	8	7	3	0	B	4	9	5	2	6	1	A
9	B	A	6	3	2	7	0	8	5	9	4	1
4	4	3	B	8	7	0	5	1	A	2	9	6
5	3	2	A	7	6	B	4	0	9	1	8	5
8	0	B	7	4	3	8	1	9	6	A	5	2
3	5	4	0	9	8	1	6	2	B	3	A	7
1	7	6	2	B	A	3	8	4	1	5	0	9
2	6	5	1	A	9	2	7	3	0	4	B	8
A	A	9	5	2	1	6	B	7	4	8	3	0
7	1	0	8	5	4	9	2	A	7	B	6	3
6	2	1	9	6	5	A	3	B	8	0	7	4
B	9	8	4	1	0	5	A	6	3	7	2	B

It contains the Hadamard matrix below between  $T_1$ IP and Q. The pattern of 1s in the two matrices is identical.

	8	7	3	0	B	4	9	5	2	6	1	A
1											1	
A												1
5								1				
6										1		
9							1					
4						1						
2									1			
3			1									
B					1							
8	1											
7		1										
0				1								

The two matrices  $E(P,Q)$  and  $E(IP,Q)$  contain all the Hadamard matrices between the row-class of P and the row Q.<sup>5</sup>

<sup>4</sup> These identities are explained in the document *Comparing Twelve-Tone rows:Hadamard Matrices*.

<sup>5</sup>Hadamard matrices between P and other members of the row-class of Q don't need to be generated, because if we know the permutational pattern of KP to Q, the same permutational pattern relates P to  $K^{-1}Q$ . (K is any combination of  $T_n$ , I, and/or R;  $K^{-1}$  is the inverse of K, also a combination of  $T_n$ , I, and/or R.

## Invariance Matrices

There are two types of *invariance matrices*; the *T-matrix* and the *I-matrix*. These are also known respectively as difference and sum tables or matrices.

The E(P,Q) matrix listing the intervals from P to Q is the *T-matrix of P and Q* and the E(IP,Q) matrix listing the intervals from IP to Q is the *I-matrix of P and Q*.

It should be clear that the T-matrix is a difference matrix since the body of the matrix contains the ordered intervals from P to Q and intervals are differences. While the E(IP,Q) matrix is a difference matrix such that cell  $E_{m,n}$  contains the difference between  $IP_m$  and  $Q_n$ , it is also a sum matrix such that cell  $E_{m,n}$  contains the sum between  $P_m$  and  $Q_n$ . We can show this by constructing an I-matrix from the sums of  $P_m$  and  $Q_n$  using the same rows as above for P and Q.

P is written to the left side of the I-matrix and Q is written on top of the matrix (as in a E(P,Q) or T-matrix), but the entries of the cells are the sums of the pcs in the rows. The body of this I- or sum matrix is identical to the body of the E(IP,Q) matrix (that is the T-matrix of IP and Q) shown on the last page.<sup>6</sup>

Here is the I-matrix for IP and Q. (The “+” shows it is an I-matrix or sum matrix).

+	8	7	3	0	B	4	9	5	2	6	1	A
0	8	7	3	0	B	4	9	5	2	6	1	A
3	B	A	6	3	2	7	0	8	5	9	4	1
8	4	3	B	8	7	0	5	1	A	2	9	6
7	3	2	A	7	6	B	4	0	9	1	8	5
4	0	B	7	4	3	8	1	9	6	A	5	2
9	5	4	0	9	8	1	6	2	B	3	A	7
B	7	6	2	B	A	3	8	4	1	5	0	9
A	6	5	1	A	9	2	7	3	0	4	B	8
2	A	9	5	2	1	6	B	7	4	8	3	0
5	1	0	8	5	4	9	2	A	7	B	6	3
6	2	1	9	6	5	A	3	B	8	0	7	4
1	9	8	4	1	0	5	A	6	3	7	2	B

### The T-matrix between a row P and itself

In order to generate the T-matrix for a row and itself, we make an E(P,P) matrix, that is, an E(P,Q) matrix, where P and Q are the same row. On the next page is the T-matrix for our row P = <038749BA2561>.

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<sup>6</sup> The significance of the sums is that if  $T_n I_a = b$ , then  $n - a = b$  and  $a + b = n$  ( $n$  is the index number of the inversion operator  $T_n I$ ). Note also that if  $T_n a = b$ , then  $n + a = b$  and  $b - a = n$  ( $n$  is the interval from  $a$  to  $b$ ). In both cases  $a$  and  $b$  would be pcs in the twelve-tone rows generating the invariance matrices, and the  $ns$  would be in the cells of matrices.

A T-matrix for P and itself is the same as a row table for the row P. The reason is that the tables houses the intervals between P and P, thus these intervals read in the columns from top to bottom correspond to the pcs in the  $T_n$  forms of P and the intervals read left to right correspond to the pcs in the  $T_n$  forms of P. Here is the T-matrix for P and itself.

T-matrix (row table)

	0	3	8	7	4	9	B	A	2	5	6	1
0	0	3	8	7	4	9	B	A	2	5	6	1
3	9	0	5	4	1	6	8	7	B	2	3	A
8	4	7	0	B	8	1	3	2	6	9	A	5
7	5	8	1	0	9	2	4	3	7	A	B	6
4	8	B	4	3	0	5	7	6	A	1	2	9
9	3	6	B	A	7	0	2	1	5	8	9	4
B	1	4	9	8	5	A	0	B	3	6	7	2
A	2	5	A	9	6	B	1	0	4	7	8	3
2	A	1	6	5	2	7	9	8	0	3	4	B
5	7	A	3	2	B	4	6	5	9	0	1	8
6	6	9	2	1	A	3	5	4	8	B	0	7
1	B	2	7	6	3	8	A	9	1	4	5	0

Thus the properties of a T-matrix for a row and itself are same as for a row table: (1) the main diagonal contains zeros, the interval between the row aligned with itself; (2) the entries symmetrically disposed around the main diagonal are inversions of each other. (If  $E_{i,j} = x$ , then  $E_{j,i} = -x \pmod{12}$ ).

Since the T-matrix is an  $E(P,P)$  matrix, the zeros form a Hadamard matrix for a row and itself. Likewise, the 1s form a Hadamard matrix for  $T_1P$  and P, and, in general, the matrices  $H(T_nP,P)$  are formed by the patterns of ns.

Below we list all 12 of the embedded Hadamard matrices in the row table of P. The reader can examine these matrices at leisure to detect permutational relations among the P and  $T_nP$  rows.

Some of these Hadamard matrices are related. Examine the Hadamard matrices for  $T_5P$  and P, and  $T_7P$  and P; that is,  $H(T_5P,P)$  and  $H(T_7P,P)$ . The 5s in  $H(T_5P,P)$  can be flipped around the main diagonal into the positions of the 7s in the  $H(T_7P,P)$ .<sup>7</sup> In general,  $H(T_nP,P)$  is transformed into  $H(T_{-n}P,P)$  by flipping the first matrix around the main

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<sup>7</sup>This results from the fact—explained in the document oh Hadamard matrices—that  $H(P,Q)$  is the same as  $H(KP,KQ)$  and that  $H(P,Q)$  flipped around the main diagonal is the same as  $H(Q,P)$ . In this case, K is  $T_7$ , the inverse of  $T_5$ .  $H(T_5P,P)$  is the same as  $H(T_7T_5P,T_7P)$  or  $H(P,T_7P)$ ; then we exchange P and  $T_7P$ , which produces  $H(T_7P,P)$  with the body of the  $H(T_5P,P)$  matrix flipped around the main diagonal with 7s rather than 5s.



diagonal into the second. This also explains why the  $H(T_0P,P)$  and  $(T_6P,P)$  matrices are symmetric around the main diagonal; they flip into themselves.

Hadamard matrices embedded in the T-matrix for P and itself:

038749BA2561			038749BA2561			038749BA2561		
0	0		1		1	2		2
3	0		4	1		5		2
8	0		9	1		A		2
7	0		8	1		9	2	
4	0		5		1	6		2
9	0		A		1	B	2	
B	0		0	1		1		2
A	0		B		1	0	2	
2	0		3	1		4	2	
5	0		6		1	7	2	
6	0		7	1		8	2	
1	0		2		1	3	2	

038749BA2561			038749BA2561			038749BA2561		
3	3		4	4		5		5
6		3	7	4		8	5	
B		3	0	4		1		5
A		3	B		4	0	5	
7	3		8	4		9	5	
0	3		1		4	2		5
2		3	3	4		4	5	
1		3	2		4	3	5	
5		3	6		4	7	5	
8	3		9	4		A		5
9	3		A		4	B	5	
4	3		5		4	6		5

<u>038749BA2561</u> 6                     6 9         6 2             6 1                 6 A              6 3   6 5                  6 4         6 8   6 B              6 0   6 7         6	<u>038749BA2561</u> 7         7 A             7 3   7 2                  7 B              7 4         7 6                  7 5                  7 9         7 0   7 1                  7 8         7	<u>038749BA2561</u> 8         8 B              8 4          8 3   8 0   8 5                  8 7         8 6                  8 A              8 1                  8 2          8 9         8
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<u>038749BA2561</u> 9          9 0   9 5                  9 4          9 1                  9 6                  9 8         9 7         9 B              9 2                  9 3   9 A              9	<u>038749BA2561</u> A              A 1                  A 6                  A 5                  A 2                  A 7         A 9          A 8         A 0   A 3   A 4         A B              A	<u>038749BA2561</u> B              B 2                  B 7          B 6                  B 3   B 8   B A              B 9                  B 1                  B 4          B 5                  B 0   B
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In the discussion of the properties of the row matrix in the document *The Row Matrix*, it was shown that the INT(P) and INT(IP) are found on the diagonals immediately to the right and left, respectively, of the row matrix's main diagonal. This follows from the properties of the E(P,Q) matrix, whose diagonals house the intervals between P and Q (as well as P and RQ) in various alignments. Since the row table is the same as E(P,P) the alignments are between P and itself.

Let us align P with itself shifted one position to the left.

038749BA2561 = row P  
038749BA2561 = row P  
35B952B4317 = intervals from row P to row P shifted 1 position to the left,  
i.e., from  $P_n$  to  $P_{n+1}$ .

Now these intervals are the same as those in the INT(P), which registers the intervals from  $P_n$  to  $P_{n+1}$ . So the diagonal to the immediate right of the main diagonal on the row

matrix lists the INT(P) as well as the intervals between the alignment of P with P shifted one position to the left..

Now we align P with itself shifted one position to the right and take the intervals likewise.

038749BA2561 = row P  
 038749BA2561 = row P  
 97137A189B5 = intervals from row P to row P shifted -1 position to the left.

These intervals are from  $P_n$  to  $P_{n-1}$ . This series of intervals is the inversion of the intervals above in the INT(P). The inversion of the INT(P) is the INT(IP). So the diagonal to the left of the main diagonal of the row matrix lists the INT(IP) as well as the intervals between the alignment of P with P shifted one position to the right..

In general,

*the xth diagonal to the right of the main diagonal of the T-matrix between P and P (the row matrix) contains the  $INT_x(P)$  as well as the intervals from row P to itself shifted x positions to the left.*

It was noted that the E(P,Q) matrix houses the series of intervals between P and RQ on its secondary diagonal and that the diagonal x positions to the right of the secondary diagonal list intervals between P and RQ shifted x positions to the right. By substituting P for Q in the last sentence, we see that these secondary diagonals in the row matrix show intervals between various alignments of P and its retrograde. To show this, we align P and RP.

038749BA2561 = row P  
 1652AB947830 = row RP  
 139262A6539B = intervals from row P to RP

The secondary diagonal of the row matrix lists the series from top to bottom. In the discussion of the row matrix, we noted that the secondary diagonal listed the intervals between  $P_n$  and  $P_{B-n}$ ; this is the same as the intervals between P and RP aligned pc to pc.

Here is P aligned with RP shifted 3 positions to the right.

038749BA2561 = row P  
 1652AB947830 = row RP  
 6283034A6 = intervals from row P to RP shifted 3 positions to the right.

The interval series is found 3 positions to right of the secondary diagonal of the row matrix on page 8.

All of the series of intervals between P and RP in any alignment have RI symmetry, due to the symmetry of a row table around the main diagonal; the main diagonal bisects the secondary diagonals.

### The I-matrix between a row P and itself

The I-matrix is the same as E(P,IP) matrix, but we write P to the left and above the matrix and consider the body of the matrix to hold the sums of the pairs of pcs from P to itself. Here is the I-matrix for our row P.

	0	3	8	7	4	9	B	A	2	5	6	1
0	0	3	8	7	4	9	B	A	2	5	6	1
3	3	6	B	A	7	0	2	1	5	8	9	4
8	8	B	4	3	0	5	7	6	A	1	2	9
7	7	A	3	2	B	4	6	5	9	0	1	8
4	4	7	0	B	8	1	3	2	6	9	A	5
9	9	0	5	4	1	6	8	7	B	2	3	A
B	B	2	7	6	3	8	A	9	1	4	5	0
A	A	1	6	5	2	7	9	8	0	3	4	B
2	2	5	A	9	6	B	1	0	4	7	8	3
5	5	8	1	0	9	2	4	3	7	A	B	6
6	6	9	2	1	A	3	5	4	8	B	0	7
1	1	4	9	8	5	A	0	B	3	6	7	2

The I-matrix has different properties from T-matrix: (1) the main diagonal has two instances of the even numbers from 0 to A; (2) the matrix is symmetric around the main diagonal.

If we align IP with P, we obtain a series of intervals as shown.

09458312A76B = row IP  
 038749BA2561 = row P  
 064286A84A02 = intervals from row IP to row P.

This series is found on the main diagonals of the I-matrix for P and itself, because this matrix is also a T-matrix between IP and P (or a E(IP,P) matrix). So the main diagonal gives the intervals from IP to P. Similarly the other diagonals on the I-matrix give the intervals between IP and P aligned in various ways.

For the reasons just stated, the secondary diagonal of the I-matrix gives the intervals between IP and RP.

09458312A76B = row IP  
 1652AB947830 = row RP  
 191928829191 = intervals from row IP to row P.

This series is palindromic, invariant under R.

And in general, the I-matrix houses the series of intervals between IP and RP shifted x positions to the right on the diagonal x positions to the right of the secondary diagonal.

Since the secondary diagonals are bisected by the main diagonal, all of the series of intervals between IP and RP in any alignment have retrograde symmetry, due to the symmetry of a row table around the main diagonal.

The following are the 12 Hadamard matrices (H(T<sub>n</sub>IP,P) for each n) embedded in the I-matrix of P with itself. Again, the reader can examine them for interesting permutational relations between T<sub>n</sub>IP, and P. Each Hadamard matrix is different unlike the Hadamard matrices with in the T-matrix for P with itself.

Since T<sub>n</sub>I is its own inverse, the matrix H(T<sub>n</sub>IP,P) the same as H(P,T<sub>n</sub>IP) and, therefore each H(T<sub>n</sub>I.P) matrix is symmetric around its main diagonal.<sup>8</sup>

Hadamard matrices embedded in the I-matrix for P and itself:

038749BA2561	038749BA2561	038749BA2561
0   0	1   1	2   2
9   0	A   1	B   2
4   0	5   1	6   2
5   0	6   1	7   2
8   0	9   1	A   2
3   0	4   1	5   2
1   0	2   1	3   2
2   0	3   1	4   2
A   0	B   1	0   2
7   0	8   1	9   2
6   0	7   1	8   2
B   0	0   1	1   2

<sup>8</sup> H(T<sub>n</sub>IP,P) = H(T<sub>n</sub>IT<sub>n</sub>IP,T<sub>n</sub>IP) = H(P, T<sub>n</sub>IP)

038749BA2561			038749BA2561			038749BA2561		
3	3		4	4		5		5
0	3		1		4	2		5
7		3	8	4		9		5
8		3	9		4	A		5
B			0	4		1		5
6			7		4	8		5
4		3	5			6		5
5			6		4	7		5
1			2		4	3		5
A		3	B		4	0		5
9			A		4	B		5
2			3		4	4		5

038749BA2561			038749BA2561			038749BA2561		
6			7	7		8		
3	6		4		7	5		8
A			B		7	0		8
B		6	0	7		1		8
2			3	7		4		8
9		6	A		7	B		8
7			8		7	9		8
8		6	9		7	A		8
4			5		7	6		8
1			2		7	3		8
0	6		1		7	2		8
5			6		7	7		8

038749BA2561			038749BA2561			038749BA2561		
9		9	A		A	B		B
6			7		A	8		B
1			2		A	3		B
2		9	3		A	4		B
5			6		A	7		B
0	9		1		A	2		B
A			B		A	0		B
B		9	0		A	1		B
7			8		A	9		B
4		9	5		A	6		B
3			4		A	5		B
8			9		A	A		B

## Summary

The matrix of intervals from one row to another, first called the  $E(P,Q)$  matrix and later the T-matrix (between P and Q) embeds the twelve Hadamard matrices  $H(T_n P, Q)$  and holds in its diagonals the intervals between various alignments of P and Q or RQ. The T-matrix made from P and itself is the row table. The properties of the row table are therefore derived from the properties of a T-matrix except that the properties between P and a copy of itself in the T-matrix were conflated into the properties internal to P itself in the row table. The I-matrix between P and Q is a sum matrix, but is actually the T matrix between  $IP$  and Q. It is mainly useful when Q is P, since it then displays relations between  $T_n IP$  and P or RP.

### Other uses of invariance matrices.

The T and I matrices can be used to show the permutational and content relations between any two ordered sets of pcs, of any cardinality, possibly with duplications. The theory of (Stravinskian) rotational arrays is easily modeled by invariance matrices.

These matrices can be used to generalize the common-tone theory from transposition to any operations. Under this interpretation they can answer the question: how many common pcs are there between P and  $T_n KQ$ , where P and Q are pcsets. K is *any* operation, and not only I or M.

These matrices can be deployed in other aspects of pitch (not only pitch-class) and other musical dimensions, such as contour, time, and timbre and upon transformations in addition to entities.