

Sets, Scales and Rhythmic Cycles

A Classification of Talas in Indian Music

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Theory and practice in Indian music is based on two fundamental musical structures. Partially ordered cycles of pitch classes called *ragas* provide a pitch syntax for improvisation and composition, while various cycles of beats called *talas* form the basis for Indian music's rhythmic organization. Since a myriad of ragas and talas are handed down by tradition, some of them hundreds of years old, much of Indian music theory is devoted to their classification, thereby providing some coherence and unity to Indian music as a whole. Classification is accomplished by studying the properties and characteristics (called *lakshana*-s) of various ragas and talas grouping them according to their similarities and differences. This enterprise has its corollaries in many aspects of Western theory, and especially in atonal theory, where the set classes of the standard, twelve equal-tempered pitch-classes are grouped and compared according to various formal similarity relations.

This paper contributes to Indian theory by using some theoretic tools from Western music to classify talas in *Carnatic*, or "South Indian" classical music. The paper also contributes to Western theory by suggesting ways to parse any collection of rhythmic structures into equivalence and similarity classes. Specifically, I use concepts and methodology originally introduced to study pitch in order to study temporal concepts and entities. Milton

Babbitt's time-point system of 1964 is perhaps the first and best known example of this approach, followed by more recent work by Richard Cohn, David Lewin, Robert Morris, John Rahn, and others.¹ I also apply work in (diatonic) scale theory to study rhythmic cycles. Scale theory, pioneered by John Clough and Gerard Meyerson, uses mathematical number theory to investigate the properties of cycles of pitch-classes.²

While this is hardly the first paper studying Indian music using Western theoretical concepts,³ I feel it is still necessary to remind the reader that I am studying a non-Western music without explicitly invoking its cultural contexts. Thirty years ago this enterprise would have fallen within the province of ethnomusicology, but current research in ethnomusicology is a decidedly anthropological undertaking, emphasizing music's roles in or as culture, leaving the discussion of music sound and structure (as such) to theorists within a particular music-culture. It is therefore incumbent upon theorists in one music culture investigating another culture's music to examine their motives and methodologies so as to be reasonably sure that they do not inadvertently misrepresent or misinterpret the other culture's musics to their own ends. Fortunately, as I have argued elsewhere, there has been enough contact between Indian classical music and Western art music to minimize misunderstanding and unmask ethnocentricity at least in the scholarly domain. Besides, there are quite a few striking cultural similarities between Western and Indian—especially South Indian—art musics.⁴ More pertinently however, as I have pointed out above, the classification of (abstract) musical entities is one of the basic research programs of Indian musicology,⁵ so the use of Western music theory methodology to that end need not be considered a foreign intrusion into the indigenous and traditional

study of Indian music. Indeed, some of our theoretical tools have been independently developed or borrowed by Indian scholars for their own purposes. And in the West, Oliver Messiaen and other composers have literally employed talas in their music since the 1940s.

Talas, as I implied above, are rhythmic cycles. Talas are distinguished by a number of factors. First, they may vary in *length*, that is, number of beats, from 1 to 128 beats. Second, talas are usually partitioned into parts called *anga*-s. For instance, the tala *Saara* has a length of eight beats divided into three successive angas of 3, 2, and 3 beats. Talas of the same length but different anga durations are not identical. For instance the tala *Adi* has eight beats divided into three angas, but the successive anga durations are 4, 2, 2 beats. Other Indian criteria for differentiating talas will be discussed below.

I will examine two traditionally defined collections of Carnatic talas. I have chosen these two systems since they contrast in many obvious ways. The two collections are the 35 *sulaadi talas*,⁶ one of three tala systems currently in use,⁷ and a collection of older talas, the *astottara sata talas*⁸ or the *108 talas*, most of which are musically extinct. The sulaadi talas are discussed in almost all accounts of rhythm in Indian music, while the list of 108 is presently available in only one English source. There are many other ancient talas mentioned and/or disclosed in the classical Indian music literature;⁹ perhaps the best known are the 120 *deci talas*,¹⁰ enumerated in S'aarngadeva's *Sangitaratnaakara* (1240 C.E.).¹¹ By contrast, the source for the 108 talas is only perhaps 200 years old.¹²

[Example 1 goes here.]

Ex. 1 catalogs the 35 sulaadi talas in tabular form to display their generative scheme. Seven basic *sapta* talas, given in the left side of the table, are made out of three of six basic rhythmic building blocks known as *shadanga*-s. The shadanga-s used are *laghu*, *drutam*, and *anudrutam*; they are listed with their Indian symbols below the table. The drutam and anudrutam consist of two beats and one beat, respectively, while the laghu is of variable duration, of 3, 4, 5, 7, or 9 beats. The five different laghu durations provide five classes (*jati*-s) for each of the seven sapta talas providing 35 talas in all. The jati-s are given on the top of the table. The 35 cells of the tala table each hold one of the 35 talas. It is important to note that the three shadanga-s are not only of different durations but of different qualities; the laghu is counted and is therefore digital and silent, whereas the drutam and anudrutam involve claps and waves and are analogue. For example: the tala named *Jhampa tala misra jati* (also called *Misra jhampa*) is located on the fourth row, fourth column of the tala table of Example 1. It is of 3 angas and 10 beats, having anga durations of 7 + 1 + 2; or clap, 6 finger counts, clap, clap, wave. (Modern practice expands the 35 talas to 175 by allowing each beat to be divided into 3, 4, 5, 7, and 9 equal parts called *gati* or *nadai*.)

It will be convenient in the sequel to have short abbreviations of the names of the sulaadi talas. For any tala of the 35, we shall use the first letter of its basic sapta tala plus a superscript denoting its jati, that is the length of the laghu. So, for example, *Druva tala tisra jati*, in the top left cell of the tala table whose laghu is of 3 beats, will be denoted by D³.

[Example 2 goes here.]

While the sulaadi talas are systematically organized, the 108 *astottara sata tala*-s are not, having no single criterion for inclusion. In addition, there are six talas on the list of 108 that seem pair-wise identical according to the structural criteria the list provides. However, we will consider each of these three pairs to be only equivalent, not identical, since they have different names and may have had differing performance practices.¹³ Example 2 provides twelve of the 108 talas. Note that talas from the 108 will be usually listed by number, not name. Unlike the sulaadi set, all six shadanga-s are employed and the laghu is invariably set to 4 beats.¹⁴ The example shows the six shadanga-s which may remind you of Western conducting gestures. The reader will have probably also noted that the length of the shadanga-s are powers of 2, except for the shadanga of 12 beats, *plutam*. The talas in Ex. 2 were chosen to exemplify the range of similarity and difference found in the entire set. The shortest and longest talas are given (lengths 2 and 128). (The 128 beat tala *Simhanandana*, number 33 is still performed today as a test or demonstration of a master musician's rhythmic prowess.) Talas 36, 40, and 41 can be associated because their anga durations enumerate all the compositions of two 2s and a 4.¹⁵ Two of these talas, 36 and 40, are related by inclusion in the 4 2 2 4 pattern of *Ata* tala, number 72. Like *Ata*, some of the talas are their own retrogrades or retrogrades of their rotations (for instance, *Kaarika* (98) and *Gaja lila* (17)).

Incidentally, the name of tala 36, *Turanga lila*, is the title of Messiaen's huge orchestral piece completed in 1948. This name also identifies a different tala in the 120 deci talas. In regard to the names of talas, the reader might have noticed that the names of talas 71 and 72, *Eka* and *Ata*, are the same as two

of the basic sapta talas in the sulaadi system, and tala 40 (of the 108), *Prati Taala*, has the same anga structure as the sulaadi tala *Tripura tala caturasra jati*, the most frequently used tala of the 35, usually called *Adi*. This overlap of name and form is a situation all-too-frequently found in Indian music theory.

Having briefly surveyed the 35 and 108 tala systems, we are now ready to apply topics in atonal and scale theory to tala structure. As many theorists have noted, there is an isomorphism between pitch and time that allows us to study temporal events with the same tools that we use in atonal pitch and pitch-class theory.¹⁶ The isomorphism maps a pitch to a time-point and an unordered pitch interval to a duration between time points. We will henceforth call time-points beats. So sets of pitches map to sets of beats. In addition, pitch transposition by interval n corresponds to the shift of beats n beats in time.¹⁷ Pitch inversion corresponds to taking the inverse of beats (with respect to some beat designated as zero); as a consequence, the order of durations between beats is retrograded. We apply the same sorts of mappings between pitch-classes and cycles of time-points. A pitch-class maps to a timepoint-class or beat-class; and an unordered pitch-class interval (or interval-class) corresponds to a duration between beat-classes. Just as the structure of pitch-classes is cyclic, the structure of beat-classes is cyclic. Pitch-class transposition corresponds to shifting a set of beat-classes around the cycle of beat-classes and pitch-class inversion corresponds to taking the inverses of the beat-classes mod the length of the cycle.

Since most of atonal music theory deals only with the 12-pitch-class system, the isomorphism is to a cycle of 12 beat-classes and as such is equivalent to

Babbitt's time-point system. But there is no conventional, psychoacoustic, or cognitive need to limit ourselves to only 12 beat-classes, so the isomorphism can be generalized to map cyclic systems of n pitch-classes to cyclic systems of n beat-classes. Since the 12 pitch-class system of standard atonal theory has been formally defined, it is easy to adapt it to systems of any number of pitch-classes. For instance, the concept of set class is usually defined without reference to the number of pitch-classes in a pitch-class system. Thus in any system of n pitch or pitch-classes, a set class is a set of pitch-class sets related by transposition and inversion. The corresponding temporal definition defines a set class as the set of sets beat-classes related by temporal shift, retrograde, or both.

Using the isomorphism, we consider a cycle of timepoint-classes to be a temporal interpretation of an unordered pitch-class set, in a mod- K universe. Given a tala X with z angas whose successive durations are $\langle a \ b \ c \ \dots \ n \rangle$, the successive beat classes of X are: $X_0 = 0$, $X_1 = X_0 + a$, $X_2 = X_1 + b$, $X_3 = X_2 + c$, etc., $X_{(z-1)+n} = X_0$. All durations and beat-classes are reckoned mod- K , where K is the length of X , the sum of its successive durations. Taking tala 48 from Ex. 2, *Jaya*, for illustration, the top of Example 3 shows its successive anga durations as $\langle 4 \ 8 \ 4 \ 4 \ 2 \ 2 \rangle$, $K = 24$ and its beat-classes are $\langle 0 \ 4 \ 12 \ 16 \ 20 \ 22 \rangle$.

In this discussion of the pitch to time isomorphism I have used the proper term "beat-class" for the informal term "beat" that I used when discussing talas before, and I shall sometimes revert to "beat" for "beat-class" below. In any case, I shall not use "beat" (as defined three paragraphs above) to mean "time point," in the rest of this paper.

[Example 3 goes here.]

If a tala's anga durations are cyclicly permuted, its adjacent beat-classes are shifted by a constant number of beat-classes wrapping around mod-K. If the order of anga durations is reversed, the tala's beat-classes find their inverses mod-K. Thus, talas whose anga durations are retrogrades and/or cyclic-permutations of each other form a subset of the beat-class set class that contains the tala. The subset may be defined as all the shifts and shifted-retrogrades of the tala's (beat-class) set class that contain beat-class zero. We shall denote this subset by the term *tala class*.¹⁸ The cardinality of a tala class is at most twice the number of angas or durations. Example 3 shows the tala class defined by tala Jaya. It has twelve members.¹⁹

Let us now begin to classify the 35 and 108 talas defining equivalence relations as we proceed.

[Example 4 goes here.]

Talas are deemed *length equivalent* if they have the same number of beats. Length equivalence partitions the corpus of talas into *length classes*. Example 4 shows the 35 length classes formed by the members of both the 35 and 108 talas. The example shows some significant differences between the two tala systems. The 35 talas are fairly evenly distributed from lengths 5 to 12, while the set of 108 is much more sparsely dispersed over the entire range from 2 to 128 beats; moreover, it has relatively large length classes of talas, of 12, 16, 20, 24, 32, and 48 beats. These lengths are all multiples of

4—perhaps because the laghu of the 108 talas is always 4 beats long—and, except for 20, these length numbers all have many divisors. The 35 talas' lengths cover all the primes from 3 to 29 except 19. In contrast, the 108 cover only 7 primes.

[Example 5 goes here.]

The members of a length class can be divided into *partition classes*.²⁰ Talas that have anga durations that are permutations of each other are members of the same partition class.²¹ For instance, we noted that talas 36, 40, and 41 in Example 2 have anga durations that consist of two 2s and one 4. These are members of the same partition class denoted as $2^2 4$ in "power" or "multiset notation." Example 5 shows all 19 members of the length class 24 for the 108 talas. The talas are divided into 12 partition classes. For instance talas 28, 42, 48, and 68 are members of the partition class $2^2 4^3 8$.

The partition classes can be likewise divided into tala classes which were defined to contain talas that have anga durations that are rotations, retrogrades, or both of each other. Example 5 shows tala class membership by lower-case letters. We see that the four members of partition class $4^2 8^2$ are divided into two tala classes labeled c and d. Tala class c contains anga durations consisting of a pair of 4s followed (or preceded) by a pair of 8s, wrapping around, while tala class d contains anga durations alternating 4 and 8.

[Example 6 goes here.]

Example 6 shows how the talas are divided into length classes, partition classes, and tala classes. Note that among the 108 talas, the number of distinct talas is 105 due to the three pairs of equivalent talas mentioned above. Once again the two tala collections are differentiated. While the 108 talas are divided into different cardinalities of length, partition, and tala classes, the 35 talas each represent a unique partition class and tala class.

Aside from this tidy scheme of embedded equivalence classes for classifying the talas according to length, partition, and tala class, there are other modes of equivalence that cut across this taxonomy. Grouping talas into anga classes is one such mode. Talas with the same number of angas are members of the same *anga class*.

[Example 7 goes here.]

Upon inspecting Example 7, we see that the sulaadi talas encompass only 4 different anga classes, those of one to four angas. This follows from the systematic way the 35 talas are organized; for instance, the Matya, Jhampa, and Triputa sapta talas have three angas each. The second column of the example additionally shows which of the 7 sapta talas of the 35 are in which anga class. The number of talas in the anga classes is somewhat skewed toward the higher anga classes. This is the opposite of the 108 talas, which are heavily skewed toward the lower anga classes. The four largest anga classes of the 108 talas, 3 to 6, contains 78 talas, ranging from 3 to 48 beats. This means the anga classes intersect with many different length classes.

[Example 8 goes here.]

Like the length classes, the anga classes can be partitioned into sub-classes—in this case, the *d-seg classes*. The notion of a d-seg was introduced by Elizabeth West Marvin to extend the concepts of pitch contour into the temporal domain.²² A d-seg class is the set of all series of d durations whose successive elements are ordered according to a series S of integers coding the relative lengths of duration. The series S denotes the class and is called a d-seg. Example 8 shows the anga durations of the talas T⁵, 40, and 104. While their length and angas are different, the talas have the same number of angas in the pattern of one duration followed by two shorter but equal durations. We denote the relative pattern of durations with the series S <100> which is the d-seg. The d-seg class <100> of Example 8 contains all the talas whose anga durations can be described by the d-seg <100>.

Marvin has pointed out that the 35 talas are organized by d-seg class inclusion.²³ The jati-s of each sapta tala form the members of the same d-seg class. This is because the laghu is always longer than the drutam or anudrutam. The bottom of Example 8 documents this observation.

[Example 9 goes here.]

D-seg classes are fairly numerous among the 108 talas with no class having more than 4 members. Example 9 presents the 18 talas of the 108 system that are members the 3 anga class; these are grouped into 9 d-seg classes. Note that the d-segs exhaust all permutations of one 1 and two zeros, two 1s and one zero, and trivially, three zeros.. Of the two d-segs that include 0, 1, and 2, the 2 is last.

There are many other equivalences that can be drawn to relate talas, but we will only cite two more. First, the number of and different types of shadanga-s in a tala is of interest. In the 35 talas, the anudrutam of one beat is used only in the Jhampa talas, while the drutam of two beats occurs in all but the Eka talas. The laghu occurs in all 35 talas and among the sapta talas with more than one anga, the laghu occurs in three of them once, in two others twice, and in the remaining on three times. Once again we witness a symmetry in the 35 talas. Of the six shadanga-s in the 108 talas, the kaakapadam of 16 beats only occurs once, in the longest tala of 128 beats. The shadanga of one beat is relatively rare, occurring as the last anga in 14 of the 18 talas in which it is found.

[Example 10 goes here.]

The last equivalence involves possible self-symmetry of a tala's anga durations. Many talas have invariant anga durations under either rotation or rotation and retrograde. These invariances are images of T and RI invariances among cycles of pitch-classes (under the pitch-to-time isomorphism).²⁴ If a tala possesses such invariance, the number of members of its tala class is less than maximum, which we said was twice the number of angas in the tala. Note also that members of equivalence classes based on invariance are not obliged to have the same length or number of angas. Thus invariance relates talas from different length and anga classes.

There are some characteristic patterns of numbers in anga duration series that indicate invariance. These are shown on the top of Example 10. The

left-most column labeled "pattern" shows a pattern of letters. For instance, the third row has a series of Xs. The next column labeled multiset abbreviation writes the patterns in the first column in power notation. So the series of Xs on the third row left, is written as X^a to the ath power, indicating there are a Xs in the pattern. The name column takes a salient feature of the multiset abbreviation and uses it as a name for the pattern. On the third row, the exponent a of the power notation is used to denote the pattern of a Xs. Finally, the right column identifies the kind of invariance possessed by the pattern in the left. Again on the third row, the series of a Xs has both rotational and retrograde/rotational symmetry (denoted by the word "all"), for if we rotate, reverse, or rotate and reverse the series of Xs, the pattern is invariant. The fourth row indicates that a series of Xs interrupted by one Y will have retrograde/rotational invariance. We denote this case with the name a_1 which stands for the exponents in the multiset abbreviation. The next to last pattern is simply a series of three or more different symbols set equal to S. The pattern is S repeated a times. The last pattern is a retrograde invariant set of symbols (in which the letters in parentheses may be omitted without disrupting the retrograde). The bottom of Example 10 gives some examples of talas that have these invariances.

Each of the different invariance patterns on the top of Example 10 forms a unique equivalence class; this provides seven different ways to classify talas. And, in turn, we may group the seven classes into three super-classes: first, the s pattern which provides rotational invariance; second, the four patterns which indicate retrograde-rotation invariance; and third, the two (trivial) patterns, 1 and a, which permit both rotational and retrograde/rotational

symmetry. And of course, we may group all talas into two classes; those that have invariance and those that don't.

Far from being properties of only a few talas, invariance patterns are found among the anga durations in most of the talas in both collections. Among the 35 talas, 30 have invariance. This occurs because six of the seven sapta talas have invariance patterns which are inherited by their jati-s. In particular, the a1 pattern occurs 15 times, and the 11, ab, and 1 patterns each occur five times. All of these invariances are of the retrograde-rotation type. The Jhampa talas are the only talas of the 35 that do not have an invariance. More than two-thirds of the 108 talas have invariance. Unlike in the 35 talas, all seven types of patterns are to be found in the 108. However, non-trivial rotational invariance (the s type) is relatively rare, possessed only by 7 talas, one of which is given in Ex. 10. Talas 98, 67, and 29 have an interesting relation. The anga durations of 98 are 2 2 2 2 1 which is repeated twice to generate the anga durations of tala 67, 2 2 2 2 1 2 2 2 2 1, and thrice to generate tala 29. In this way, all the rest of the rotationally invariant talas of the 108 are generated by concatenating simpler ones also in the collection.²⁵

Up to now we have been grouping talas into equivalence classes. This kind of classification is crisp; either a tala shares some feature with another or it does not. Next I arrange talas in an ordered list according to some feature that admits degrees of difference. Talas that have the feature to the same degree share the same position on the list and can be considered equivalent; talas that differ slightly with respect to the feature—that is, they are adjacent, or almost adjacent on the list—can be considered similar.

[Example 11 goes here]

One feature for use in ordering talas is *fragmentation*. We have already grouped talas into partition classes. All talas in a particular partition class have the same fragmentation. We use the partition P as the input to a function that yields the fragmentation of the partition. Fragmentation varies between 0 and 1 and is a measure of the uniformity of a distribution—the higher the fragmentation, the more even the distribution. We calculate the fragmentation of a partition of the number N into z parts using the following formula where $\{p_1, p_2, \dots, p_z\}$ is the partition P. Note: $N = p_1 + p_2 + \dots + p_n$ and $PAIRS(s) = \frac{s^2 - s}{2}$.

$$FRAG(P) = 1 - \frac{\sum_{k=1}^z PAIRS(p_k)}{PAIRS(N)}$$

[Example 11 goes here.]

The talas 36, 40, or 41 have anga durations that form the partition $2^2 4$ of the number 8. The formula determines their fragmentation as .7142857.²⁶ These talas and their shared fragmentation are shown among others in Example 11.

While each member of a partition class has the same fragmentation, fragmentation is not limited to partition class membership. For instance, many of talas in Example 11 have the same fragmentation but are of different numbers of beats and angas. Talas 79 and 81 share a fragmentation of .4 but the two do not share the same number of beats. Likewise, there are

three talas with a fragmentation of .73333..., all of which have a different number of angas. In sum, 61 talas share fragmentation values to form 20 equivalence classes according to fragmentation; each of these classes have two or more members.

The values of fragmentation for the two tala systems range from .3272727 to 1. Each tala system encompasses the entire range, overlooking the fact that the 108 excludes the lowest value. In the 35 talas, there are only two cases of shared fragmentation values: J⁴ and T⁵ share .6666... and the Eka talas (Eⁿ) share 1. As might be expected, the two systems share very few fragmentation values; only three (.6666..., .7333..., .787878...) are shared.

The second feature that I use to induce an ordering of the talas depends on the pitch-to-time isomorphism; talas (cycles of beat-classes) are the temporal counterpart of scales (cycles of pitch-classes). We may therefore borrow concepts and methodology from scale theory to suggest other realms of tala classification. A particularly suggestive concept in scale theory is that of *Heteromorphism*. Heteromorphisms were devised by Jay Rahn to study the seven-pc mod-12 set classes as scales.²⁷ Incidentally, heteromorphisms can be used also to suggest why some of the 72 *melakarta* scales of Carnatic Music are considered "artificial" (deemed *vivadi melas*) by some Indian musicologists.²⁸ In any case, one can define and count the number of heteromorphisms in a tala by treating it as if it were a scale.

There are three kinds of heteromorphism: difference, ambiguity, and contradiction. We say a scale S has first-order *differences* if its steps are not all of the same size. Only scales that are generated by one interval, like the

whole-tone scale, have no first-order differences. S has a first-order *ambiguity* if there is an interval of a leap (a non-adjacent interval) that is equal²⁹ to any scale step (adjacent interval). For instance, in the harmonic minor scale, the leap from the first to third degrees is equal to the step from the 6th to 7th degrees, so the scale has a first-order ambiguity. Similarly, if S has a leap that is less than a scale step, we say S has a first-order *contradiction*. The Carnatic *mela* (parent-scale) "Jhalavarali," <C, D-flat, E-double-flat, F-sharp, G, A-flat, B, C> has a first-order contradiction since the leap from C to E-double-flat, 2 semitones, is less than the step of 4 semitones from E-double-flat to F-sharp. It also has a first-order ambiguity formed by the presence of the step of 4 semitones from E-double-flat to F-sharp and the leap from A-flat to C, also of 4 semitones.

[Example 12 goes here]

Similarly, we can enumerate the first-order heteromorphisms in a cycle of time-points, where scale step intervals correspond to the successions of durations between adjacent beat-classes, and leaps in scales correspond to durations between non-adjacent beat-classes. We use a Stravinskian rotational array, constructed in the usual way mod the tala length, to quickly and efficiently locate the heteromorphisms. Example 12 shows the rotational array for tala 48, Jaya tala. (Some numbers in the matrix are highlighted or noted in various ways for the purpose of this discussion.). As in all rotational arrays of this type, this left-most column is called the zeroth column and contains 0s. The next column to the right holds the tala's successive anga durations written from top to bottom; this is column 1 and it contains the first-order (that is, adjacent) durations between the talas beat-classes. Here the column 1 displays Jaya's anga duration series. The next column, number

2, holds the second-order durations between the tala's beat-classes. These are durations between beat-classes two order positions apart. And in general, the nth column holds the nth-order durations between beat-classes n positions apart.

Now, for any rotational array constructed from a particular tala, the number of different numbers minus 1 in column 1 gives the number of first-order differences in the tala. Looking at column 1 of the rotational array of Jaya, we find three different numbers so there are two differences and the number 2 is written underneath column 1 on the row headed #DIF. As for ambiguities, every number z on the array in a column to the right of column 1 that is equal to a number on column 1 represents a first-order ambiguity. Similarly, a z in a column numbered higher than 1 that is less than a number on column 1 represents a first-order contradiction. On the array for Jaya, the first-order ambiguities are shown with asterisks (the three 8s with asterisks are each equal to the 8 in column 1, and the 4 with asterisks is equal to the three 4s on column 1 and therefore takes three asterisks); thus there are six ambiguities shown by the 6 printed under column 1 in the row marked #AMB. I show the first-order contradictions with carets (the 6s and the 4 in column 2 are less than the 8 in column 1). Once again below column 1, the number of first-order contradictions is shown in the #CNT row. Thus Jaya has 2 differences, 6 ambiguities and 3 contradictions of the first order.

Before continuing with the enumeration of heteromorphisms, we should stop to consider what difference, ambiguity, and contradiction mean in the temporal domain. Difference simply denotes the number of different durations between adjacent beat-classes in a rhythmic cycle minus 1. A

first-order ambiguity indicates that there is a span of time between two non-adjacent beat-classes that is equal to the duration between two adjacent beat-classes. A first order contradiction means that there is a span of time between two non-adjacent beat-classes that is less than the duration between two adjacent beat-classes. The presence of ambiguities and contradictions can make it difficult to hierarchize a rhythmic cycle or sequence, for instance, to suggest a sense of meter. Consider a series of equally spaced beats. It is easy to hierarchize (in one's mind) the succession of beats into a series of beats twice as slow, three times as slow, etc. One reason for the ease is that the durations between the beats on any higher and therefore slower-going level are all longer than the durations between the beats themselves. Now consider a series of unequally spaced beats that has a first-order ambiguity or contradiction. At least one duration E between every other (or n th) beat will be less or equal to a duration D between two successive beats. Thus we cannot solely depend on durations between non-adjacent beats to establish a given hierarchic level. Given enough ambiguities, and especially contradictions, inferring temporal hierarchic levels perceptually may be near to impossible.

Returning to counting heteromorphisms and Example 12, the n th-order differences, ambiguities, and contradictions are indicated below column n in a rotational array. The total number of each type of heteromorphisms is given on the left. With ambiguities and contradictions, the total is simply the sum of the n th-order heteromorphisms. With differences, the total is cardinality of the union of all non-zero columns minus 1.

Four other rotational arrays are given in example 12. Tala J³ has an ambiguity, but no contradictions. Tala 107 has the converse situation, and talas M⁵ and 63 have neither ambiguities nor contradictions.

[Example 13 goes here]

Example 13 tallies the number of talas with respect to ambiguity and contradiction. The 35 sulaadi talas again prove singular since ambiguities are rare and no tala in the set has both ambiguities and contradictions. Furthermore, very few of the 35 have either ambiguities or contradictions. Approximately four fifths of the 108 talas have ambiguities and/or contradictions, but tend not to have contradictions alone.

Looking again at Example 12, below each rotational array are four pieces of data that help to compare talas of different numbers of angas and cardinalities. *Ambdepth* indicates the highest order of durations that has ambiguities; this highest order of ambiguities has the number O as a limit. For a tala of length L, O is equal to $O/2 - 1$ where L is even, and $O/2 - 1/2$ where L is odd.³⁰ In the second array, of tala J³, shows only first order-ambiguities; thus the ambdepth is 1. *Adjamb*, standing for adjusted ambiguities, is the total number of ambiguities divided by Z, where Z is the total number of durations between beat-classes. Z is therefore the square of the number of angas. It is quite possible for adjamb to be larger than 1; this means there are more ambiguities than durations in the tala. The values of *contdepth* and *adjcontra* are determined for contradictions in the same ways as for ambiguities, except that the limit O is set to L for contdepth.

[Example 14 goes here]

Example 14 displays the ranges for ambdepth, adjamb, contdepth, and adjcontra.

Ambiguities and contradictions figure into another property of scales and rhythmic cycles: maximum equality of step size. Such maximally equal (ME) sets have been studied by a number of theorists, but I shall use an adaption of the definition of a maximally equal set found in Clough and Douthett's 1991 article.³¹ A *maximally equal (ME) tala* has no more than two different values in each non-zero column of its rotational array and the two values are consecutive. The last array of Example 12 shows a ME tala. Each column in the tala's rotational array (except the zeroth) show two values and each pair of values differs by 1. There are 9 ME talas from the 108 and 4 from the 35.^{32 33} With ME talas we are again within the domain of equivalence classes, for talas either have it or they do not.

ME talas are not frequent in the two systems, but we can posit other types of talas that have properties similar to ME. First we define the *distributionally equal*, or *DE talas*, after Clough, Engebretsen and Kochavi's DE sets whose intervals have the same properties.³⁴ The definition of DE is the same as ME, except the no-more-than-two distinct values in a rotational array column do not have to be consecutive. An example of such a tala is illustrated in the next to last part of Example 12.³⁵ There the pairs of values in columns 1 and 2 are not consecutive. Of course, the DE talas include the ME talas. What is "equal" about a DE tala is that at most two durations are found between n beat-classes apart, for all non-zero n. Another type of tala

are those that have *Myhill's property*, denoted by the acronym *MP*. MP talas have exactly two different values in each non-zero column of its rotational array. Thus talas with only one distinct anga duration are not MP—nor are those that have an array column with only one distinct value.

[Example 15 goes here]

Despite our addition of DE and MP talas to ME talas, there are some talas whose anga durations have as many as two values but are not DE, MP, or ME talas. These are "unsuccessful," so to speak, candidates for ME and DE. Example 15 shows that there are 88 candidates, 58 talas and 30 talas from the two tala systems; 30 of the candidates are unsuccessful.³⁶ The example goes on to show that 58 talas have the DE property, 51 have MP, 13 have ME and that 7 talas are trivial since they have only one anga. The seven DE talas that do not have the MP property are the same seven talas that have rotational invariance mentioned earlier.

There are many other properties and features of scales that can be imported into the world of talas, but I leave that to future research. Suffice it say that recent work by Norman Carey on well-formed sets and Clough and Douthett on hyper-tetrachords are of immediate interest.

Conclusions

The isomorphism of pitch to time has permitted us to use Western music theories to investigate and categorize talas. Since the pitches in Indian ragas form scales, and scales are the counterpart of rhythmic cycles under the

pitch-to-time isomorphism, both the melodic and rhythmic sides of Indian music theory might be unified by our methodology, especially by the addition of Western scale theory to Indian ways of understanding musical time. And what better use for generalized scale theory than within the temporal domain? For while applying scale theory to pitch relations has yielded some interesting results, the new microtonal systems and tuning refinements often exceed the discriminations of the finest ears. On the other hand, when scales are interpreted as rhythmic cycles, any reasonably-sized modulus provides an easily perceived basis for making and analyzing music that takes advantage of the distinctions and properties the theory provides.

In any case, our examination of these two talas systems has highlighted their differences. The system of 35 sulaadi talas was modeled by the use of d-seg classes. Most other ways of relating these talas grouped most of them together, or failed to collect them into equivalence classes at all, since each tala represented a different class under a particular equivalence definition. The seemingly unruly 108 tala system has many modes of association producing a web of relations on the basis of length, partition, tala class, d-seg, anga, invariance, shared fragmentation, and scalar properties. One can then, by successively linking talas via different types of equivalence, find a smooth path to get from any tala to any other in the system. Ambiguities and contradictions are much less prevalent among the 35 talas than among the 108. While neither tala system is closed so that all the members of a particular equivalence class are represented in the system, the members that are represented share a high degree of similitude due to the properties that permits the equivalence in the first place.

Studying the 35 and 108 talas with these and other methods not only sheds light on the possible reasons why these particular talas were grouped together, but also provides ways of parsing any collection of rhythmic structures into equivalence and similarity classes. Thus this research offers some insights into the construction of similarity-relations on temporal entities, in addition to pitch collections and scales as in previous research.

Notes

¹See: Milton Babbitt, "Twelve-Tone Rhythmic Structure and the Electronic Medium," *Perspectives of New Music* 1/1 (1964):53-61; John Rahn, "On Pitch and Rhythm: Interpretations of Orderings of and in Pitch and Time," *Perspectives of New Music* 13/2 (1975):182-203; David Lewin, *Generalized Musical Intervals and Transformations* (New Haven and London: Yale University Press, 1987); Robert Morris, *Composition with Pitch-Classes: A Theory of Compositional Design* (New Haven and London: Yale University Press, 1987); Richard Cohn, "Transpositional Combination of Beat-Class Sets in Steve Reich's Phase-shifting Music," *Perspectives of New Music* 30/2 (1992):146-177.

²See: John Clough and Gerald Meyerson, "Variety and Multiplicity in Diatonic Systems." *Journal of Music Theory* 29/2 (1985): 249-69; John Clough and Jack Douthett, "Maximally Even Sets." *Journal of Music Theory* 35/1&2 (1992): 93-174; David Clampitt and Norman Carey, "Aspects of Well-formed Scales," *Music Theory Spectrum* 11/2 (1989):187-206.

³For instance, see Clough, John, Jack Douthett, N. Ramanathan, and Lewis Rowell "Early Indian Heptatonic Scales and Recent Diatonic Theory." *Music Theory Spectrum* 15/1: 36-58.

⁴While South Indian music has its roots in the Hindu temples and villages, the music has become primarily urban since 1945. Like Western music, Carnatic music puts great emphasis on the musical personality of the composer and improviser as cultural exemplars. Indians are fond of comparing their "Trinity of musician/saints," Tyagaraja, Muthuswami Dikshitar, and Shyama Shastigal, to Mozart, Haydn, and Beethoven, who

lived at approximately the same time. As in the West, Carnatic music was originally sponsored by royal courts and religious institutions. Today it is patronized by the more prosperous classes of India.

⁵Carnatic musicology reaches back at least 400 years to the period when Hindusthani (North Indian) and Carnatic music diverged into separate musical systems, and then back more than an entire millennium to Sanskrit treatises such as the *Naatyas'aastra* or the Tamil Classic *Cilappitikaaram* (both circa 200 C.E.). Lewis Rowell *Music and Musical Thought in Early India* (Chicago: University of Chicago Press: 1992) provides an excellent overview of the history of Indian music and its scholarship. P.

Sambamoorthy, *South Indian Music* 6 vols. (Madras: The Indian Music Publishing House, 1960-69) covers the more recent history and theory of Carnatic music.

⁶ See, for instance, Bonnie C. Wade, *Music in India: the Classical Traditions* (Westwood, MA: Riverside Co., 1992) or P. Sambamoorthy, *South Indian Music, Book II*.

⁷The other two Carnatic and contemporary tala families are the so-called *chapu talas* and the *nava sandhi talas* used in temple rituals.

⁸The list with comments is on pp. 152-62 in P. Sambamoorthy *South Indian Music , Book IV*. The source text is the Sanskrit treatise *Ashtotara-tala-lakshanam*, probably written at the beginning of the nineteenth century. The manuscript is preserved at the "Sarasvati Mahal Library" in Tanjavur, Tamil-Nadu, South India. (I am indebted to N. Ramanathan for information on the history of the 108 talas.)

⁹See P. Sambamoorth, *South Indian Music , Book IV*:146-168.

¹⁰An annotated list of the 120 deci talas is found in Messiaen's *Traité de Rythme, de Couleur, et d'Ornithologie, Tomb I*. (Paris:Alphonse Leduc, 1994). Rowell provides examples of 10 deci talas in his discussion of the 120 in *Music and Musical Thought*. Both Rowell and Messiaen propose different ways of categorizing the deci talas, according to criteria similar to those used in this paper.

¹¹At present, only a portion of the entire *Sangitaratnaakara* is available in English Translation.

¹²Ramanathan believes that the treatise is not over 200 years old since the idea of collecting talas into collections of 108 is a relatively recent Tamil concept, even though many names of the 108 are names found in various collections of deci talas such as the aforementioned *Sangitaratnaakara*. Note however, many of the talas sharing the same name are not identical.

¹³The three pairs are: *Varnabhinna* (21) and *Lalita* (89); *Traysra varna* (28) and *Nandi* (68); and *Nissaaruka* (50) and *Vishama kankalla* (76).

¹⁴However, there is some question about the length of the laghu of the first five talas of the 108. Some old treatises assert that these talas, known as *marga talas*, have laghu-s of five beats or parts.

¹⁵Talas whose angas' durations are retrogrades and/or cyclic permutations of each other are members of what we will call a tala class.

¹⁶Some theorists prefer to consider the pitch/pc system and the time point/time point-class systems as two different interpretations of an uninterpreted abstract system.

¹⁷If n is positive the shift is n beats forward in time; if negative, the shift is backwards in time.

¹⁸As I pointed out before, talas 36, 40, and 41 of Example 2, are members of the same tala set-class. This class has only three members since any of its members is invariant under cyclic permutation and retrogression.

¹⁹Equivalence under retrograde, which depends on the symmetry of anga durations of a tala, could be ruled out as a meaningful transformation in Indian rhythmic theory. Since only the anudrutam, the shadanga of one beat, is its own retrograde, retrograding the other five shadanga-s would turn them into an undefined set of claps, finger counts, and waves.

²⁰Partitions are discussed in *prastaara*, the last of the ten topics in the *taala dasa praana*-s, the traditional presentation of Carnatic rhythmic theory. See P. Sambamoorthy, *South Indian Music*, Book IV : 182-4.

²¹Under the pitch-class to beat-class isomorphism, Allen Forte's concept of BIP-related pc segments is equivalent to the concept of partition class. See Allen Forte, *The Structure of Atonal Music* (New Haven: Yale University Press, 1973):63.

²²A D-seg class is the set of all series of d durations (D-segs) whose successive elements are ordered according to the same series of relative lengths of duration. See Elizabeth West Marvin, "The Perception of Rhythm in Non-Tonal Music: Rhythmic Contours in the Music of Edgard Varèse," *Music Theory Spectrum* 13/1 (1991):61-78.

²³*Ibid.*

²⁴The temporal images of the RI-symmetry among series of pitch-classes, is the cyclic version of Messiaen's "non-retrogradable rhythms." T-symmetry among pitch-classes (equivalent to Messiaen's "modes of limited transposition") have images as talas with retrograde-rotational invariance.

²⁵Tala 53 has anga durations of 8 4 12 which generates tala 43's anga durations, 8 4 12 8 4 12. Likewise for: tala 18 (4 8 12) and tala 45; tala 90 (4 8) and tala 42; tala 81 (41) and tala 78.

²⁶Here $N = 8$ and $z = 3$. The PAIRS function on the three members of the partition $p_1 = 2$, $p_2 = 2$, and $p_3 = 4$ is respectively 1, 1, and 6 which sum to 8. This is divided by $\text{PAIRS}(8)$. $\text{PAIRS}(8) = 28$. Thus the $\text{FRAG}(\{2,2,4\}) = 1 - 8/28$.

²⁷The three concepts of heteromorphism in scales—difference, ambiguity, and contradiction—were invented by Jay Rahn to study the seven-pc mod-12 set-classes as scales. See Jay Rahn, "Coordination of Interval Sizes in Seven-Tone Collections," *Journal of Music Theory* 35/1&2 1992: 33-60.

²⁸See Jay Rahn, "Structure, Frequency and Artificiality in South Indian Melas." in T. Temple Tuttle, ed., *Proceedings of the Saint Tyagaraja Music Festivals, 1978-81*. (Cleveland Ohio: The Greater Cleveland Ethnographic Museum, 1981)

²⁹To within enharmonic equivalence.

³⁰The reason for the limit O on ambdepth is that in a tala of length L an ambiguity between an n -th order interval and an $(n+k)$ -th order interval implies an ambiguity between an $(L-n)$ -th order interval and an $(L-n+k)$ -th order interval. If we were not to set O as a limit, then both ambiguities would contribute to ambdepth.

³¹Clough, John and Jack Douthett, "Maximally Even Sets," *Journal of Music Theory*, 35 (1991):93-173.

³²The ME talas are those that have laghu-s of three beats.

³³While there is no necessary relation between ME and the presence of ambiguities, the jhampa talas are not candidates for ME and are also singular

with respect to ambiguity. Of the remaining 30 candidates of the 35, only the Ata (A^n) talas are excluded from ME having three values in some of their rotational array columns.

³⁴DE scales are defined in Clough, John, Nora A. Engebretsen, and John Kochavi, "Scales, Sets, and Interval Cycles: A Taxonomy," *Spectrum of Music Theory*, forthcoming.

³⁵This tala, if asserted as a pcset in the usual mod-12 pitch-class universe, would be a member of SC (3-9)[027]. Four other trichords have DE, which, together with 3-9, comprise the inversionally invariant trichords. No trichordal set-class has ME.

³⁶ Unsuccessful candidates have more than two distinct values in at least one of their array's columns.

Ex. 1. The 35 *sulaadi talas*.

<i>Sapta Talas</i> ↓	<i>Jati-s</i> →				
	3 Tisra	4 Caturasra	5 Khanda	7 Misra	9 Sankirna
Druva	³ 0 ³ ³	⁴ 0 ⁴ ⁴	⁵ 0 ⁵ ⁵	⁷ 0 ⁷ ⁷	⁹ 0 ⁹ ⁹
Matya	³ 0 ³	⁴ 0 ⁴	⁵ 0 ⁵	⁷ 0 ⁷	⁹ 0 ⁹
Rupaka	0 ³	0 ⁴	0 ⁵	0 ⁷	0 ⁹
Jhampa	³ ∪ 0	⁴ ∪ 0	⁵ ∪ 0	⁷ ∪ 0	⁹ ∪ 0
Triputa	³ 0 0	⁴ 0 0	⁵ 0 0	⁷ 0 0	⁹ 0 0
Ata	³ ³ 0 0	⁴ ⁴ 0 0	⁵ ⁵ 0 0	⁷ ⁷ 0 0	⁹ ⁹ 0 0
Eka	³	⁴	⁵	⁷	⁹

Talas are divided into from one to four *angas* (limbs).

Angas are of three types called *shadangas*:

|ⁿ = a *Laghu* anga of n beats (clap and n–1 finger counts)

0 = a *Drutam* anga of two beats (clap, wave; or clap, backclap)

∪ = an *Anudrutam* anga of one beat (clap).

Example: the tala named *Jhampa tala misra jati* is of 3 angas and 10 beats, divided |⁷ ∪ 0 (7 + 1 + 2; or clap, 6 finger counts, clap, clap, wave). This tala is also named *Sura*.

Ex. 2. Twelve of the *astottara sata* (108) *talas*.

Tala #	name	angas	anga durations	length
17	Gaja lila	~	4 4 4 4 1	17
25	Simhavikridita	88 88 8	4 4 12 8 4 8 12 4 12	68
31	Hamsanaada	8008	4 4 12 2 2 8	32
36	Turanga lila	00	2 2 4	8
37	Simhanandana	88 8 8 0088 8 88 + 	8 8 4 12 4 8 2 2 8 8 4 12 4 12 8 4 4 16	128
40	Prati taala	00	4 2 2	8
41	Dvitiya	0 0	2 4 2	8
48	Jaya	8 00	4 8 4 4 2 2	24
71	Eka taala	0	2	2
72	Ata taala	00	4 2 2 4	12
98	Kaarika	0000~	2 2 2 2 1	9
107	Raagavardhana	800~08	8 2 2 1 2 12	19

Angas are of six shadangas:

~	anudrutam	1 beat	(clap)
0	drutam	2 beats	(clap wave)
	laghu	4 beats	(clap and 3 finger counts)
8	guru	8 beats	(clap and 7 finger counts)
 8	plutam	12 beats	(clap, left wave, right wave; 4 beats each))
+	kaakapadam	16 beats	(clap, palm up, left wave, right wave; 4 beats each)

Ex. 3. Beat class representation of the tala *Jaya*, number 48.

anga durations: 4 8 4 4 2 2
beat-classes: 0 4 12 16 20 22 (0)

K = length = the sum of anga durations = 24.

The twelve members of the Tala-class generated by Jaya

cyclic permutations of Jaya's anga durations	corresponding shifts of beat-classes of Jaya
<484422>	{0 4 12 16 20 22}
<844224>	{0 8 12 16 18 20}
<442248>	{0 4 8 10 12 16}
<422484>	{0 4 6 8 12 20}
<224844>	{0 2 4 8 16 20}
<248442>	{0 2 6 14 18 22}

cyclic permutations of retrograde of Jaya's anga durations	corresponding shifts of the inverse of Jaya's beat-classes
<224484>	{0 2 4 8 12 20}
<244842>	{0 2 6 10 18 22}
<448422>	{0 4 8 16 20 22}
<484224>	{0 4 12 16 18 20}
<842244>	{0 8 12 14 16 20}
<422448>	{0 4 6 8 12 16}

Example 4. Length classes of talas.

length	35 talas	108 talas
2		71
3	E^3	6
4	E^4	51 81
5	$R^3 E^5$	63
6	$R^4 J^3$	
7	$R^5 J^4 T^3 E^7$	
8	$M^3 J^5 T^4$	36 40 41 91
9	$R^7 T^5 E^9$	20 85 98
10	$M^4 J^7 A^3$	78
11	$D^3 R^9 T^7$	87
12	$M^5 J^9 A^4$	7 49 61 72 90 93
13	T^9	56
14	$D^4 A^5$	9 77
15		83
16	M^7	14 19 21 33 46 58 79 89 97 104 108
17	D^5	17 82
18	A^7	67 101
19		107
20	M^9	50 57 60 62 69 73 74 75 76 92 94
21		64
22	A^9	34 59
23	D^7	
24		2 5 10 13 18 26 28 42 48 52 53 54 55 68 70 80 95 103 106
27		29
28		15 65 84 96 100 102
29	D^9	
32		1 12 16 23 27 31 32 38 39 47 66 86
36		35 44 88
40		22 30
48		3 4 24 43 45 105
60		99
64		11
68		25
72		8
128		37

Example 5. Length class 24 of the 108 Talas.

tala #	anga durations	length	#angas	partition classes	tala classes
5	8 8 8	24	3	8^3	a
18	4 8 12	24	3	4 8 12	b
53	8 4 12	24	3	4 8 12	b
2	8 4 4 8	24	4	$4^2 8^2$	c
13	4 8 8 4	24	4	$4^2 8^2$	c
52	4 8 4 8	24	4	$4^2 8^2$	d
54	8 8 4 4	24	4	$4^2 8^2$	c
95	4 4 12 12	24	4	$4^2 12^2$	e
10	2 2 4 8 8	24	5	$2^2 4 8^2$	f
70	4 4 4 4 8	24	5	$4^4 8$	g
80	8 4 8 2 2	24	5	$2^2 4 8^2$	h
103	4 4 8 4 4	24	5	$2^2 4 8^2$	g
106	2 2 4 4 12	24	5	$2^2 4^2 12$	i
28	4 2 2 4 4 8	24	6	$2^2 4^3 8$	j
42	2 2 4 4 4 8	24	6	$2^2 4^3 8$	k
48	4 8 4 4 2 2	24	6	$2^2 4^3 8$	j
55	8 2 2 2 2 8	24	6	$2^4 8^2$	k
68	4 2 2 4 4 8	24	6	$2^2 4^3 8$	j
26	2 2 2 2 4 2 2 8	24	8	$2^6 4 8$	l

Example 6. Counts of various tala equivalence classes.

tala list	# of length classes	# of partition classes	# of tala classes	# of distinct talas
108 talas	31	78	83	105
35 talas	19	35	35	35

Example 7. Anga classes

anga class*	# of members in the 35 talas	# of members in the 108 talas	total # of members
1	5 (E^n)	2	7
2	5 (R^n)	4	9
3	15 ($M^n J^n T^n$)	18	33
4	10($D^n A^n$)	24	34
5		28	28
6		18	18
7		4	4
8		2	2
9		3	3
10		1	1
14		1	1
15		1	1
18		1	1
32		1	1

Anga class n includes talas having n angas.

Example 8. D-seg classes.

Talas in the same dseg class <100>.

tala #	name	d-seg	length
T ⁵	Dushkara (Khanda tripata)	5 2 2	9
40	Prati taala	4 2 2	8
104	Jhallaka	8 4 4	16

The 35 sulaadi talas and their dseg classes.

tala #	angas	d-seg class	# angas
D ⁿ	ⁿ 0 ⁿ ⁿ	<1011>	4
M ⁿ	ⁿ 0 ⁿ	<101>	3
R ⁿ	0 ⁿ	<01>	2
J ⁿ	ⁿ ≈ 0	<201>	3
T ⁿ	ⁿ 0 0	<100>	3
A ⁿ	ⁿ ⁿ 0 0	<1100>	4
E ⁿ	ⁿ	<0>	1

Example 9. Nine d-seg classes partitioning the 18 members of the 3 anga class of the 108 talas.

tala #	anga durations	d-seg	length	#angas
5	8 8 8	0 0 0	24	3
7	2 2 8	0 0 1	12	3
36	2 2 4	0 0 1	8	3
58	4 4 8	0 0 1	16	3
97	2 2 12	0 0 1	16	3
41	2 4 2	0 1 0	8	3
50	4 8 8	0 1 1	20	3
76	4 8 8	0 1 1	20	3
18	4 8 12	0 1 2	24	3
101	2 4 12	0 1 2	18	3
40	4 2 2	1 0 0	8	3
104	8 4 4	1 0 0	16	3
60	8 4 8	1 0 1	20	3
53	8 4 12	1 0 2	24	3
59	8 2 12	1 0 2	22	3
20	4 4 1	1 1 0	9	3
51	2 2 1	1 1 0	5	3
75	8 8 4	1 1 0	20	3

Example 10. Invariance patterns in anga durations.

pattern	multiset abbreviation	name	type of symmetry
X	X^1	1	all
XY	X^1Y^1	11	ret-rot
XXXXX....	X^a	a	all
XXXYYYYXX...	X^aY^1	a1	ret-rot
XXXYYYYXXXX	X^aY^b	ab	ret-rot
XYZ... = S: SSS...	S^a	s	rot
XY..(Z)..Y(X)	\sim	r	ret-rot

tala #	anga durations	pattern name
2	8 4 4 8	ab
6	4	1
8	2 2 1 4 2 2 1 4 2 2 1 4 2 2 1 4 2 2 1 4 2 2 1 4 2 2 1 4 2 2 1 4	s
14	2 2 2 2 8	a1
32	4 8 8 4 8	r
33	4 4 2 2 2 2	ab
93	2 2 2 2 2 2	a
R^n	n 2	11
M^n	n 2 n	a1

Example 11. Fragmentation of selected talas.

108 #	35 #	length	fragmentation	anga durations
	R ⁹	11	.3272727	2 9
	R ⁷	9	.3888889	2 7
79		16	.4000000	4 12
81		5	.4000000	4 1
	J ⁴	7	.6666667	4 1 2
	T ⁵	9	.6666667	5 2 2
20		9	.6666667	4 4 1
58		16	.6666667	4 4 8
104		16	.6666667	8 4 4
36		8	.7142857	2 2 4
40		8	.7142857	4 2 2
41		8	.7142857	2 4 2
	J ³	6	.7333333	3 1 2
14		16	.7333333	2 2 2 2 8
78		10	.7333333	4 1 4 1
31		20	.7822581	4 2 2 4 8
8		72	.9664570	2 2 1 4 2 2 1 4 2 2 1 4 2 2 1 4 2 2 1 4 2 2 1 4
	E ⁿ	n	1	3
6		4	1	4
71		2	1	2

Example 12. Heteromorphisms among talas.

tala 48 (Jaya): # of angas = 6 length = 24

anga durations: 4 8 4 4 2 2

beat-classes: 0 4 12 16 20 22

(rotational array)

0	4	12	16	20	22
0	8	12	16	18	20
0	4	8*	10	12	16
0	4	6^	8*	12	20
0	2	4***^	8*	16	20
0	2	6^	14	18	22

#DIF = 11 2 3 3 3 2

#AMB = 20 6 6 4 4 0

#CNT = 18 3 6 6 3 0

ambdepth = 2 contdepth = 4

adjamb = .555556... adjcontra = .500000

tala J³: # of angas = 3 length = 6

anga durations: 3 1 2

beat-classes: 0 3 4

0	3	4
0	1	3
0	2	5

#DIF = 4 2 2

#AMB = 1 0 1

#CNT = 0 0 0

ambdepth = 1 contdepth = 0

adjamb = .111111... adjcontra = 0

tala 107: # of angas = 5 length = 19
 anga durations: 2 2 1 2 12
 beat-classes: 0 2 4 5 7

0	2	4	5	7
0	2	3	5	17
0	1	3	15	17
0	2	14	16	18
0	12	14	16	17

#DIF = 11 2 2 2 2
 #AMB = 0 0 0 0 0
 #CNT = 15 6 6 3 0

ambdepth = 0 contdepth = 3
 adjamb = 0 adjcontra = .600000

tala M⁵: # of angas = 3 length = 12 (**DE tala**)
 anga durations: 5 2 5
 beat-classes : 0 5 7

0	5	7
0	2	7
0	5	10

#DIF = 3 1 1
 #AMB = 0 0 0
 #CNT = 0 0 0

ambdepth = 0 contdepth = 0
 adjamb = 0 adjcontra = 0

tala 63: # of angas = 4 length = 7 (**ME tala**)
 anga durations: 2 2 2 1
 beat-classes: 0 2 4 6

0	2	4	6
0	2	4	5
0	2	3	5
0	1	3	5

#DIF = 5 1 1 1
 #AMB = 0 0 0 0
 #CNT = 0 0 0 0

ambdepth = 0 contdepth = 0
 adjamb = 0 adjcontra = 0

Example 13. Counts of ambiguities and contradictions in talas.

	108 talas	35 talas
no ambs, no cont.	24	22
ambs, no cont.	29	3
no ambs, cont.	9	10
ambs, cont.	46	0

Example 14. Ranges of ambdepth, adjamb, contdepth, and adjcontra.

ambiguities

108 talas

35 talas

Ambdepth	# of talas	lowest adjamb	highest adjamb	# of talas	lowest adjamb	highest adjamb
0	34	.0	.0	32	.0	.0
1	26	.04	.25	3	.11	.25
2	36	.04	.69			
3	6	.41	.82			
4	3	.66	1.17			
6	1	.2.26	2.26			
8	1	2.18	2.18			
10	1	.75	.75			

contradictions

108 talas

35 talas

Contdepth	# of talas	lowest adjcontra	highest adjcontra	# of talas	lowest adjcontra	highest adjcontra
0	53	.0	.0	25	.0	.0
1	4	.11	.11	7	.11	.11
2	14	.04	.31	3	.25	.25
3	17	.08	.60			
4	11	.20	1.14			
5	2	.612	.836			
6	2	.10	1.00			
7	2	.11	.47			
12	1	2.61	2.61			
27	1	3.40	3.40			
22	1	.75	.75			

Example 15. Counts of maximally even (ME), Myhill property (MP), and distributionally even (DE) talas.

	108 talas	35 talas
candidates*	58	30
non-trivial DE	38	20 (D^n, M^n, R^n, T^n)
non-trivial MP	31	20 (D^n, M^n, R^n, T^n)
non-trivial ME	9	4 (D^3, M^3, R^3, T^3)
trivial	2 (talas 6 and 71)	5 (E^n)

*A tala that is a candidate for DE or ME has no more than two different values among its anga durations.