Some Musical Applications of Minimal Graph Cycles

In recent music theory, graphs—nodes connected by (possibly labeled) arcs or arrows—have become an important tool for modeling the structure of music, musical structures, and composition systems. Transformational networks (Lewin, 1987, 1993), compositional spaces (Morris, 1995), K-nets (Lewin, 1990, 2002), Perle cycle-sets (Perle, 1977; Headlam, 2002) , Tonnetz (Cohn, 1996, 1997), voice leading spaces (Morris, 1998), and canon graphs (Morris, 1997) are examples. This paper examines an important property of such graphs, the number and content of a graph's minimal cycles. Minimal cycles have no nonadjacent duplication of node content and they include no sub-cycles.[1] Of course, a graph may have no cycles (when it is a tree or poset), but when it does, an enumeration of its minimal cycles is a useful way to characterize the graph—especially when the graph is complex and its visual presentation is too convoluted to provide much insight into its form and properties.[2] Moreover, minimal cycles can provide ways to compare the musical entities modeled by graphs and assess their similarity.

We shall start with the construction of graphs from relations and the enumeration of the resulting graph's minimal cycles using a computer-implemented algorithm. We shall then apply the study of minimal cycles to issues in (1) twelve-tone theory, (2) K-

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[1] In other words, minimal cycles are simple cycles, not like, say, a figure eight. A minimal cycle may be included in another minimal cycle in the sense that the order of nodes in the first cycle is also in order in the second but in the context of intervening nodes.

[2] In this case, an adjacency matrix or linked list can also be used to portray the graph's features.

nets, and (3) “tune families” extracted from actual compositions or from various modal/tonal theories including Indian music.

Given a graph of nodes connected by directed arcs or arrows that may be labeled: the nodes may represent musical qualia such as pitches or pitch-classes, time points, time points, intervals and the like; the arrows represent the successions between and among the nodes; the labels may indicate the category of the successions of the graph being instances of relations such as intervals, or transformations from one node to another. Graphs may be constructed from relations. Relations are themselves (simpler, perhaps more basic) graphs such as sets of ordered sets , unordered sets, partially ordered sets, or cycles. Both relations and the graphs they construct may or not be partitioned into disconnected subgraphs.

Ex. 1 shows a graph constructed from parts of a combinatorial row design; the first hexachord of P and the last of T5IP have the same pcs but in different orders. These two ordered hexachords are the graph’s input relations and are melded together to construct the graph as shown. The example illustrates the graph’s minimal cycles.

Twelve-tone rows can be compared by taking them in pairs to form a graph and examining the minimal cycles. In Ex. 2, we see that a visual representation of a graph melding two related rows[3] and an analysis of the graph’s minimal cycles. Ex. 3 shows another row combination and its graph and sketches a method of comparing row pairs. The graphs of rows, trios, and beyond are probably too complicated to present visually, so a list of their distinct minimal cycles is a better way to understand their pc relations

[3] Even with as few as two input relations the visual representation of the graph may be quite complicated.

and/or similarity.[4] Graphs of row pairs can be used to compose music that has pair-wise pc affinity since each pc can only move to at most two other pcs. Music composed from the minimal cycles of Ex. 2 is presented in Ex. 5.

Minimal Klumpenhouwer networks (K-nets) can be constructed from relations that each contain the cycles of a different pc operation such Tn, I, or other non-standard operations. Ex. 6 shows that the minimal cycles of such a graph are K-nets. In this way all possible K-nets can be efficiently and quickly constructed.

Taking the phrases from a (melodic) composition as relations and constructing a graph, we make a "transition system" graph for the melody whose minimal cycles represent a sort of synopsis or paraphrase of the entire composition or its parts. See Ex. 6. The traditionally specified phrases of a melodic structure like an Indian raga can form a graph whose minimal cycles capture its melodic essence. If such structures specify that their "notes" have properties in addition to pitch-class such as intensity or agogic accents and/or ornamentation, these distinctions can be registered as different nodes having the same pitch or pitch-class content. Ex. 6 shows how this works on the north Indian raga Shri.

The paper will also show how minimal cycles can help organize two-partition (Morris, 1983) and voice-leading graphs. It ends with some thoughts on how minimal cycles might be implicated in models of musical cognition and memory.

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[4] The combinatorial blow up is striking even for graphs of three rows. In one case of four trichordally related derived rows (as in Webern's Opus 24) there are 1,053,692 minimal cycles of which 6 are distinct. There are methods, however, for making the cyclic enumeration of such graphs manageable.

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Ex. 1

Hexachordal Combinatoriality:

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P:     C B  G Ab Eb Db|D  Bb GbF E  A
T5IP: F F# Bb A  D  E |Eb G  B C Db Ab
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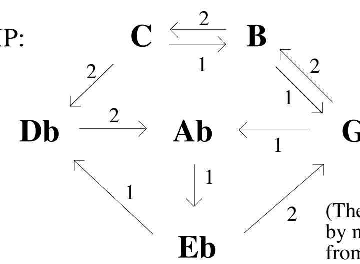
Graph derived from first hex of P and last hex of T5IP:

Input relation 1: C-B-G-Ab-Eb-Db

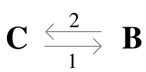
Input relation 2: Eb-G-B-C-Db-Ab



(The arcs are labeled by numbers indicating from which input relation the pcs they connect are derived.)

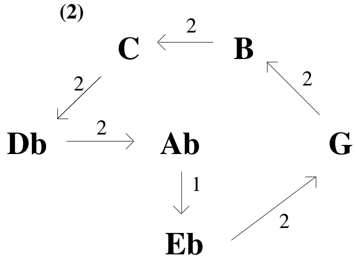
The four minimal cycles of the graph:

(1)



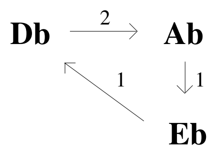
(C B -)

(2)



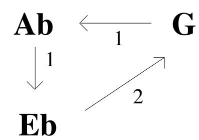
(C-Db-Ab-Eb-G-B -)

(3)



(Db-Ab-Eb-)

(4)



(Eb-G-Ab-)

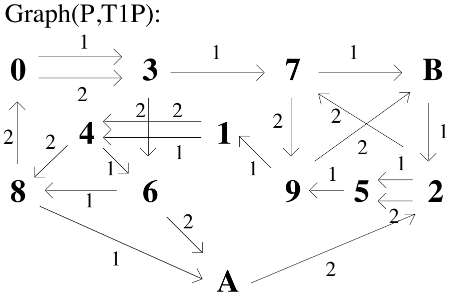
Ex. 2

A graph constructed from two related rows P and T1P.

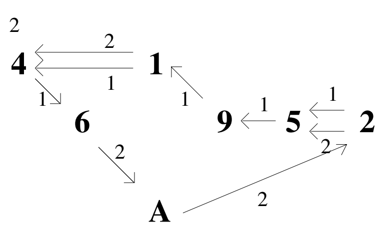
Input relation 1: T0P: 0-3-7-B-2-5-9-1-4-6-8-A

Input relation 2: T1P: 1-4-8-0-3-6-A-2-5-7-9-B

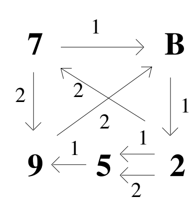
Graph(P,T1P):



minimal cycle 1: (1-4-6-A-2-5-9-)



minimal cycle 2: (2-5-7-9-B-)



This graph contains 76 minimal cycles two of which are displayed to the right. However, not all minimal cycles of a graph need be distinct. The minimal cycle 1 above actually represents four different equivalent cycles whose order of nodes is the same.[1] The four differ because there are ordered pairs of nodes connected by more than one arrow (such as nodes 1 to 4). The sum of the arrows connecting such ordered node pairs gives the number of equivalent minimal cycles.

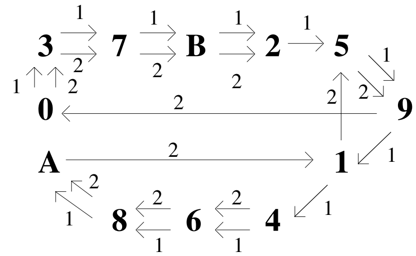
The graph contains 15 distinct minimal cycles: 3 of length 11; 1 of length 10; 1 of length 9; 4 of length 8; 3 of length 7; 2 of length 4; 1 of length 5. The number of equivalent cycles in each case varies from 4 to 8.

1. Minimal cycles form equivalence classes. The term "distinct minimal cycle" denotes a representative of the class; the class contains cycles whose nodes are in the same order but whose arc labels are different. The term "equivalent minimal cycles" denotes the class.

Ex. 3 A graph constructed from rows P and RT2IP.

Input relation 1: T0P: 0-3-7-B-2-5-9-1-4-6-8-A

Input relation 2: RT1IP: 4-6-8-A-1-5-9-0-3-7-B-2



This graph is simpler than the one in Ex. 2 because the two rows have many more shared adjacent segments. It has 3 distinct minimal cycles partitioning all 42 minimal cycles. Because there are many ordered node pairs the sets of equivalent cycles is much larger than in Ex 2. Specifically, the three distinct cycles are:

(0-3-7-B-2-5-9-) 32 equivalent cycles
(1-4-6-8-A-) 8 equivalent cycles
(1-5-9-) 2 equivalent cycles

The differences between the two graphs in regard to the number and kinds of minimal cycles suggests various similary measures between pairs of rows that form graphs. One such measure is based on the fragmentation of the set of all minimal cycles into equivalent mininal cycles. The equation for this measure and others will be discussed in the paper.

Ex. 4 Some theorems about the minimal cycles of pairs of rows.

These simple theorems and others are discussed in the paper.

1. The graph of P and RP has 11 minimal 2-node cycles, all distinct.

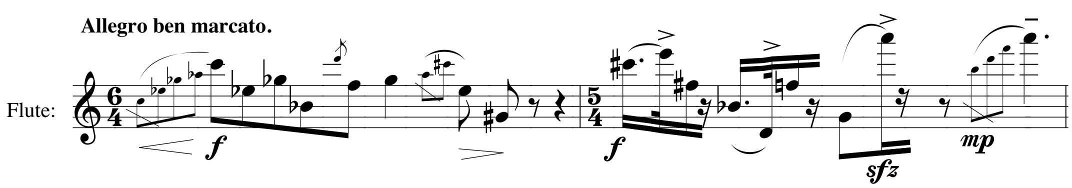
2. The graph of P and a rotation of P has 1024 minimal cycles of 11 nodes, all equivalent .

3. If rows P and Q begin with the pc a and end with pc b, their graph's minimal cycles exclude pcs a and b.

4. If the last pc of P is the same as the first of Q and vice versa, their graph's minimal cycles include all 12 pcs.

5. Combinatorial row pairs have fewer distinct minimal cycles and generally smaller minimal cycle classes than row pairs that are not combinatorial.

Ex. 5 Music based on the minimal cycles of the rows in Ex. 2.



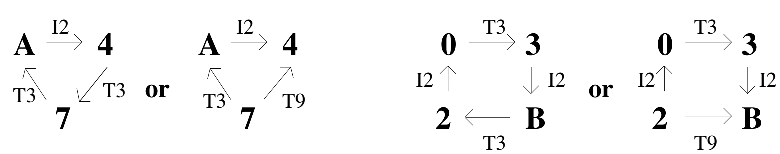
Ex. 6 Generating K-net via minimal cycles.

Input relation 1: cycles of T3: 0-3-6-9- 1-4-7-A- 2-5-8-B-

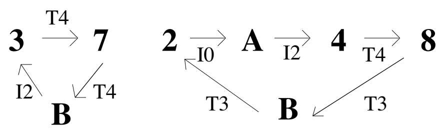
Input relation 2: cycles of T2I: 0-2- 1- 3-B- 4-A- 5-9- 6-8- 7-

Here the input relations are the cycles of various operators, each of which are disconected graphs; the constructed graph's minimal cycles are K-nets whose arcs are labeled by the operations rather than the input relation number.

Three minimal cycles = K-nets:

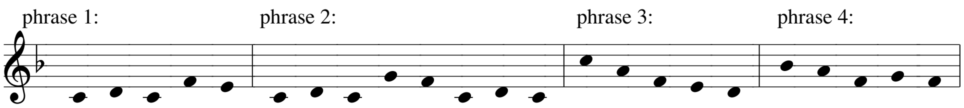


Ex. 6 continued

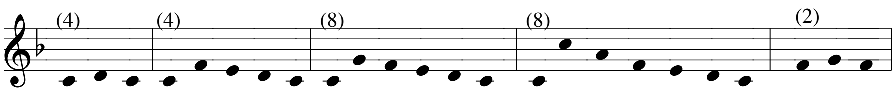


The two K-nets to the left were generated from minimal cycles for the operations T3, T4, T0I, and T2I. There are 2373 minimal cycles equalling 2373 distinct K-nets with these operations. Equivalent minimal cycles are K-nets with the same pcs in the same configuration but with different operator labels.

Ex. 7 The minimal cycles of "Happy Birthday."

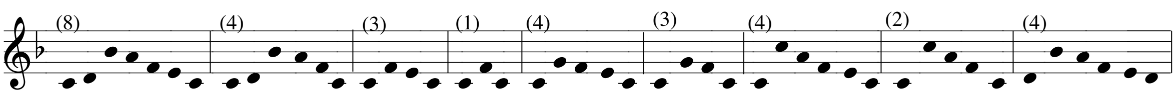


Taking each phrase as a different input relation, the distinict minimum cycles are:



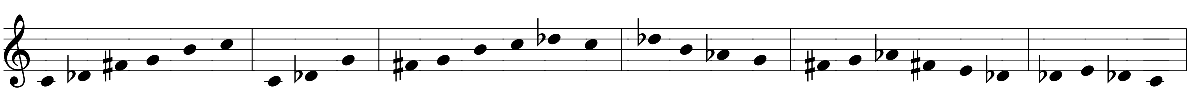
(The parenthisized numbers give the number of equivalent minimal cycles.)

Taking the entire tune as a single input relation, the distinct minimum cycles (plus the ones above) are:

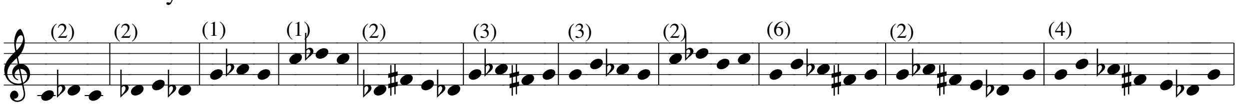


Ex. 8 The minimal cycles of the north Indian raga Shri.

These are the basic phrases of the the raga Shri according to Rao,van der Meer, Harvey(1999).



The minimal cycles:



These minimal cycles have be presented so they begin on end on the raga's salient tones (in order of importance: C, G, Db). Thus a cycle can be used to embellish one of these tones.