

Comparing Twelve-Tone Rows: Hadamard Matrices

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Any two rows are permutations of each other. Thus we can compare them according to the way a permutation changes one row into the other. In this section we present a geometric method to show how two rows are permutationally related using a *Hadamard matrix*. A Hadamard matrix is a square of numbers, with the numbers limited to 1 or 0.

We construct the matrix from two rows we will call P and Q. It is a 12 by 12 square such that the P row is written on the left of the matrix and the Q row is written above the matrix and. We name a particular Hadamard matrix H(P,Q), with P being vertical row and Q being horizontal row.¹

The 144 *cells* or *positions* of a Hadamard matrix are given a 0 or 1 based on the following rule:

If the pc in the P row to the left of the cell and the pc in the Q row above a cell are identical, we write a 1 in the cell. Otherwise the we write a 0 in the cell.

Below is a Hadamard matrix for the rows:

P = <038749BA2561>

Q = <8730B495261A>

For example, there is a 1 on the cell in the third row and first column of the matrix because pc 8 is both above and to the left of the cell. The lowest, right-hand cell is a 0 because there is A above it and a 1 to the left.

| | 8 | 7 | 3 | 0 | B | 4 | 9 | 5 | 2 | 6 | 1 | A |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| B | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| A | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

¹ H(Q,P) is a different matrix from H(P,Q) since in H(Q,P), row Q is the vertical row and P is the horizontal row.

We are interested in the pattern of the 1s in the matrix. It is a bit difficult to see this pattern among all the 0s, so we write the matrix without the 0s and cell partitions, as shown next.

| | 8 | 7 | 3 | 0 | B | 4 | 9 | 5 | 2 | 6 | 1 | A |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | 1 | | | | | | | | |
| 3 | | | 1 | | | | | | | | | |
| 8 | 1 | | | | | | | | | | | |
| 7 | | 1 | | | | | | | | | | |
| 4 | | | | | | 1 | | | | | | |
| 9 | | | | | | | 1 | | | | | |
| B | | | | | 1 | | | | | | | |
| A | | | | | | | | | | | | 1 |
| 2 | | | | | | | | | 1 | | | |
| 5 | | | | | | | | 1 | | | | |
| 6 | | | | | | | | | | 1 | | |
| 1 | | | | | | | | | | | 1 | |

To understand how to interpret the pattern of 1s, we need consider any pair of 1s in the matrix.

Contiguous unordered sets shared by rows P and Q in H(P,Q).

If the 1s are diagonally adjacent in the matrix, the pcs above the 1s in row Q and the pcs to the left of the 1s in row P are the same and adjacent in both Q and P.

For example, consider the 1s in the first two columns of the matrix. They are diagonally adjacent in the matrix, and the pcs 7 and 8, above and to the left of the 1s, are adjacent in both rows Q and P. Similarly for the 1s on the lowest rows of the matrix, they correspond to the pcs 1 and 6, which are adjacent in both rows. Note that the pc pairs 0 and 3, 4 and 9, and 2 and 5, are adjacent in both rows and diagonally adjacent in the matrix.

Thus, by looking for diagonally adjacent matrix 1s, we can easily find pairs of pcs that are next to each other in both rows.

We can extend this examination further, if we look for patterns of 1s that will fill a square region of the matrix, such that there is 1 in each row and column of the region.

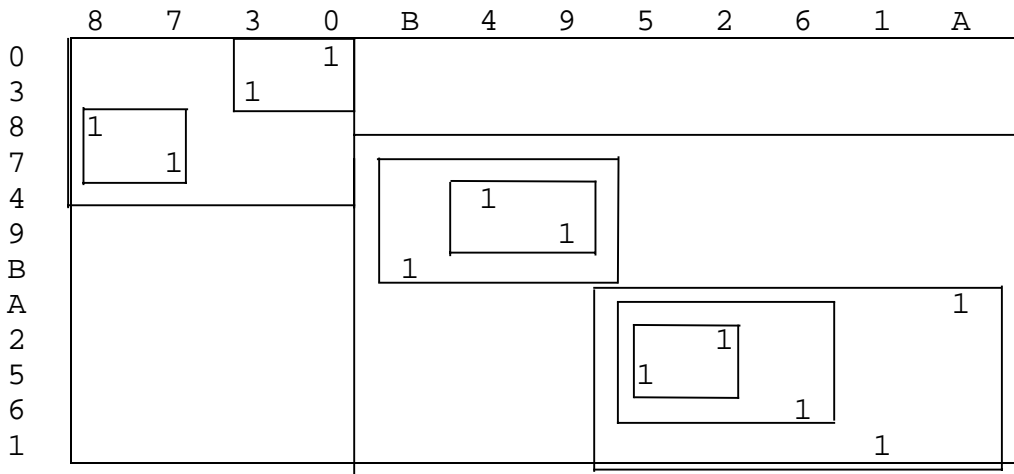
For instance, we see that the three 1s in the matrix rows 5-7 and matrix columns 5-7 form a 3 by 3 square region that has 1s in all of its rows and columns. The pcs to the left in P and above in Q in the region are the same (although in a different order) in P and Q; the pcs are 4,9 and B. Thus the set {49B} is located in contiguous positions in both rows.

The first four pcs of both rows hold different orderings of the set {0378}. This is shown in the matrix since below Q and to the left of P form we have a square region in matrix rows 1-4 and matrix columns 1-4 that has 1s in each row and column.

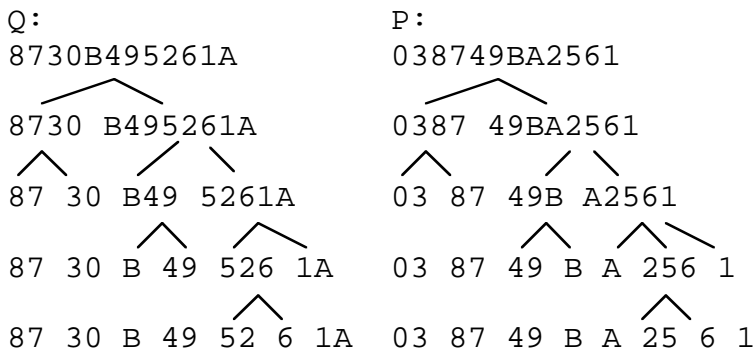
We show these two squares on the copy of the matrix below.

| | 8 | 7 | 3 | 0 | B | 4 | 9 | 5 | 2 | 6 | 1 | A |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | 1 | | | | | | | | |
| 3 | | | 1 | | | | | | | | | |
| 8 | 1 | | | | | | | | | | | |
| 7 | | 1 | | | | | | | | | | |
| 4 | | | | | | 1 | | | | | | |
| 9 | | | | | | | | | | | | 1 |
| B | | | | | 1 | | | | | | | |
| A | | | | | | | | | | | | |
| 2 | | | | | | | | | 1 | | | |
| 5 | | | | | | | | 1 | | | | |
| 6 | | | | | | | | | | 1 | | |
| 1 | | | | | | | | | | | 1 | |

When we locate all the square regions in the matrix, we find some of them overlapping or embedded in others. See the next page.



Each box shows a set that is expressed adjacently in both rows. The pattern of embedding of sets in the rows is shown by the tree diagram below.



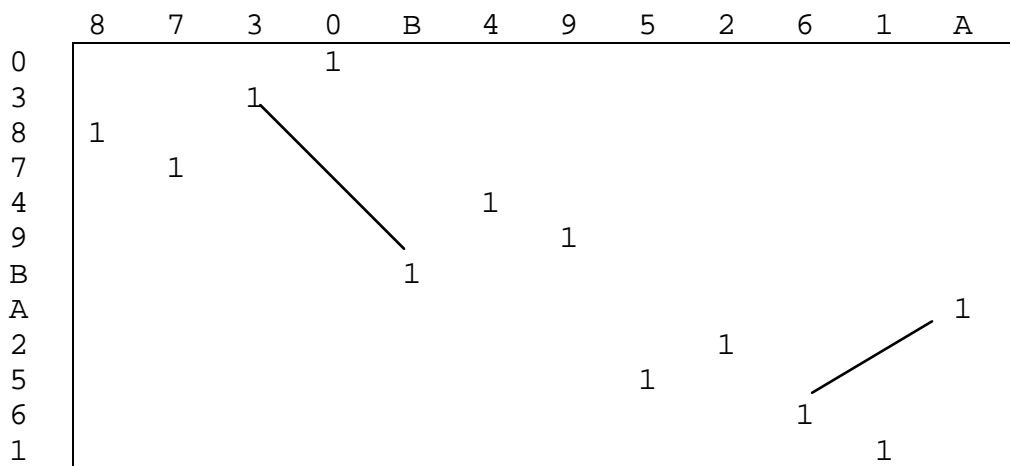
Ordered sets (adjacent or not) shared by rows Q and P.

The matrix can also show that the rows P and Q share certain ordered sets of pcs.

If two pcs a and b are in the same order in both rows, the pattern of 1s corresponding to the pcs in the matrix will be in a down-and-to-the-right direction.

If two pcs a and b are in the reverse order in the two rows, (that is the order of pcs is <a, b> in one row, and <b,a> in the other), the pattern of 1s corresponding to the pcs in the matrix will be in an up-and-to-the-right direction.

We illustrate this below:

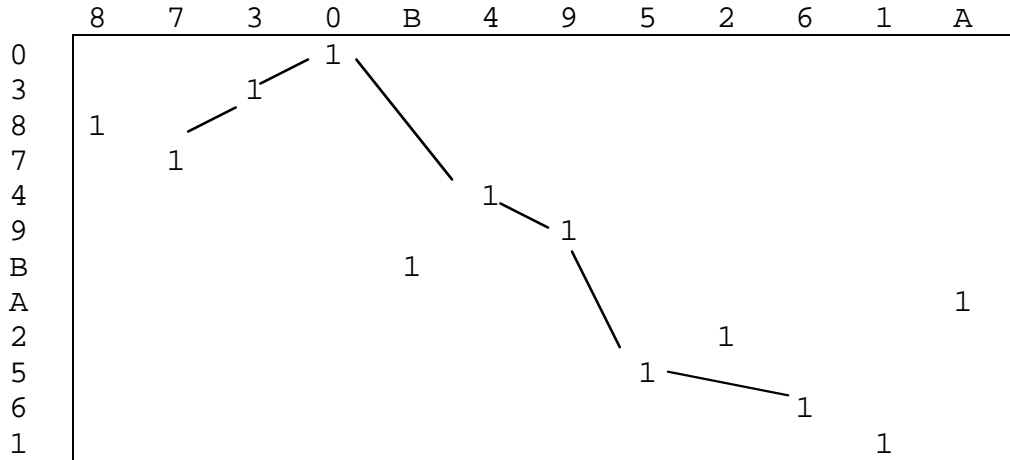


The pcs 3 and B are in the same order in the two rows. The pattern of 1s is down and to the right, as shown.

The pcs A and 6 are in that order in row P but in reverse order (6 then A) in Q. The corresponding 1s in the matrix are in an up and to the right pattern.

We can generalize this also. If we find a series of 1s in the matrix that moves down and to right, those pcs are in the same order in both rows; if the series is up and to the right, the pcs are in reversed order in the two rows.

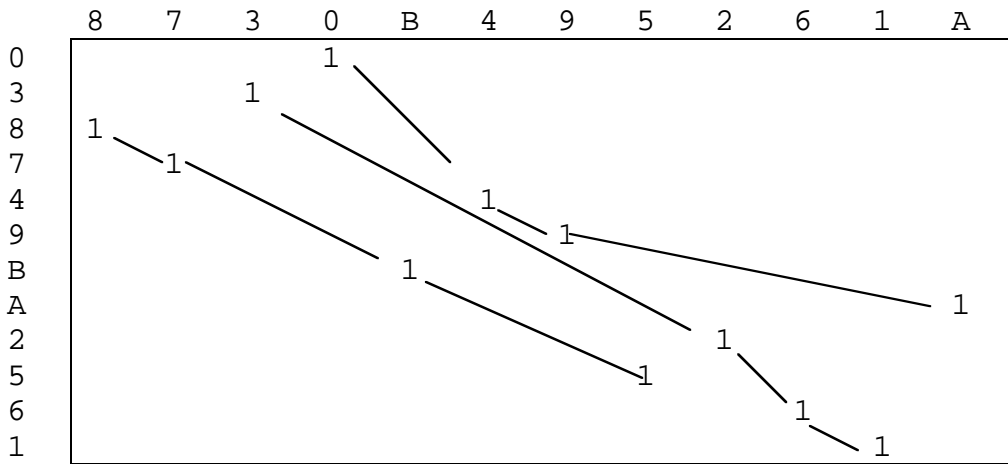
For instance, in the next copy of the matrix the lines between the 1s involve the corresponding pcs 0,4,9,5,6 down and to the right and pcs 7, 3, 0 up and to the right. This means the ordered set <04956> is shared by both rows, and the ordered set <730> is in order in one row and in reverse order in the other—in this case it is in order in Q and in reverse in P.



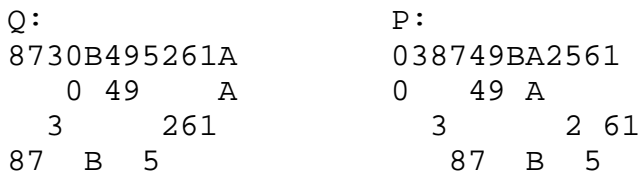
The following diagram illustrates this.

| | |
|--------------|--------------|
| Q: | P: |
| 8730B495261A | 038749BA2561 |
| 0 495 6 | 0 49 56 |
| 8730B495261A | 038749BA2561 |
| 730 | 03 7 |

The next example shows that both rows can be partitioned into three four-pc ordered sets.



The following diagram shows this.



If we think of the three ordered sets as “voices,” the rows each represent a different way of aligning the same three voices.

It is possible to find other ways to partition both rows into three identical ordered sets of equal or unequal lengths; one just connects the 1s in the matrix into three distinct lines all moving down and to the right in various ways. Some or all lines of pcs might be up and to the right, indicated shared ordered sets between RQ and P.

Now, if Q and P in H(P,Q) are from the same row-class, then they are related by a twelve-tone operator in addition to the permutational relation shown by the Hadamard matrix; then there would be *two* different ways in which they are related: 1) by sharing contiguous unordered sets and/or ordered sets; 2) by being related by a twelve-tone operation, which means the two rows share the same or retrograde series of identical or inverted intervals. This would also provide a reason to choose Q and P out of their row-class for use in composition, rather than some other pair.²

² Of course, any operation K on the two rows would change the rows but preserve their permutational relation as shown in the matrix. Thus rows KQ and KP would be permutationally related in the same way as Q and P. Thus H(P,Q) = H(KP,KQ). If K is some combination of T_n, I, and/or R, then Q, P, KQ, and KV are members of one and only one row-class.

Thus, there is motivation for making Hadamard matrices for all the pairs of rows in a row-class. In the next section on invariance matrices, we show that only two matrices will be needed to be constructed to geometrically illustrate the permutational relations among all pairs of rows in a row-class.

Operations on Hadamard matrices

We can transform a Hadamard matrix under operations such as horizontal retrograde and vertical retrograde. This will involve changes in the rows that generate these transformed matrices.

For instance, if we horizontally retrograde $H(P,Q)$, result is $H(P,RQ)$, a Hadamard matrix generated by the rows P and RQ.

Similarly, if we turn the matrix upside down—transform under vertical retrograde—the new matrix is $H(RP,Q)$

If we do both horizontal and vertical retrograde together to $H(P,Q)$, we have $H(RP,RQ)$. The result is the original matrix rotated around its secondary diagonal. This is shown by the next example, using our rows $P = \langle 038749BA2561 \rangle$ and $Q = \langle 8730B495261A \rangle$.

| | A | 1 | 6 | 2 | 5 | 9 | 4 | B | 0 | 3 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 1 | | | | | | | | | | |
| 6 | | | 1 | | | | | | | | | |
| 5 | | | | 1 | | | | | | | | |
| 2 | | | | | 1 | | | | | | | |
| A | 1 | | | | | | | | | | | |
| B | | | | | | | | 1 | | | | |
| 9 | | | | | | 1 | | | | | | |
| 4 | | | | | | | 1 | | | | | |
| 7 | | | | | | | | | | | 1 | |
| 8 | | | | | | | | | | | | 1 |
| 3 | | | | | | | | | | 1 | | |
| 0 | | | | | | | | | 1 | | | |

If we invert $H(P,Q)$ around its main diagonal, the new matrix is $H(Q,P)$ that is, the Q and P rows exchange. The example on the next page shows $H(Q,P)$.

| | 0 | 3 | 8 | 7 | 4 | 9 | B | A | 2 | 5 | 6 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | | | 1 | | | | | | | | | |
| 7 | | | | 1 | | | | | | | | |
| 3 | | 1 | | | | | | | | | | |
| 0 | 1 | | | | | | | | | | | |
| B | | | | | | | 1 | | | | | |
| 4 | | | | | 1 | | | | | | | |
| 9 | | | | | | 1 | | | | | | |
| 5 | | | | | | | | | | 1 | | |
| 2 | | | | | | | | | 1 | | | |
| 6 | | | | | | | | | | | 1 | |
| 1 | | | | | | | | | | | | 1 |
| A | | | | | | | | 1 | | | | |

The reader can compare these matrices to the original matrix on page 2.