The Complement Theorem

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In "Bob's Atonal Theory Primer," we stated the complement theorem as follows:

Complement Theorem: For pcsets A and A′, the ICV of A′ is a transformation of the ICV of A: for each entry in the ICV of A except the entry for ic6, add k (for the entry of ic6, add k/2). k = 2a – 12, where a is the number of elements in A. (NB: k = a – (12-a)).

Corollary (the Hexachord Theorem): If A is a hexachord, then k = 0, so complementary hexachords have the same ICV.

We will prove the theorem in an abstract way and interpret the result so it applies to pcs and intervals.

Consider example 1a: 12 points are placed in a space divided into two sectors 1 and 2 separated by a boundary. The number of points in sector one is P_1; the number of points in sector two is P_2.

(1) $12 = P_1 + P_2$ (and $P_2 = 12 – P_1$)

Now let us connect all the points with arrows so that each point p_n is connected to exactly two other points such that p_n is connected by an arrow to p_m and p_l is connected to p_n by an arrow for all points p_n, p_m, p_l. See example 1b. p_m may equal p_l as in example 1c. Examples 1d to 1e show the points of 1a connected by arrows in two different ways. The arrows connect the points into one or more cycles. In each of these examples n_1 is the number of arrows that connect the points in sector 1; n_2 is the number of arrows connecting points in sector 2; n_0 is the number of arrows that connect points from sector 1 to 2 or sector 2 to 1.

(2) $12 = n_1 + n_2 + n_0$.

Now examine example 2. In example 2a two arrows connect four points; in sector 1 p_1 connects to p_2 and in sector 2 p_3 connects to p_4. The arrows do not cross the boundary between sector 1 and 2. In example 2b, the same four points are reconnected so that arrows cross the boundary: p_1 connects to p_3 and p_4 to p_2. This changes the values of n_1, n_2 and n_0 without changing P_1 and P_2; both n_1 and n_2 are diminished by 1 and n_0 is increased by 2.[1]

(3) In the entire space of 12 points and arrows, as we reconnect pairs of points as indicated above, for each pair of points n_0 increases by 2 while n_1 and n_2 both decrease by 1 and P_1 and P_2 remain invariant.

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[1] It should be clear that we must change *pairs* of arrows; otherwise some points will either be without arrows pointing to or away from them, or have more than one arrow pointing to or away.

Now imagine the case where there are no points connected across the boundary between sectors 1 and 2; that means $n_0 = 0$. Ex. 3 shows such a situation. So

(4) if $n_0 = 0$, then $P_1 = n_1$ and $P_2 = n_2$.

Let (4) be the case. When we change n_0 to x by reconnecting points, from (3) we have

(5) $n_1 = P_1 - x/2$ and $n_2 = P_2 - x/2$.

(Note that x has to be even.)[2]

From (2) and (5) we derive:

(6) $12 = n_1 + (P_2 - n_0/2) + n_0$.

We solve (6) for n_0.

From (1), $12 = n_1 + (12 - P_1 - n_0/2) + n_0$

$P_1 = n_1 + n_0/2$

(7) $n_0 = 2(P_1 - n_1)$

Now from (2) and (7) we have

$12 = n_1 + n_2 + 2(P_1 - n_1)$.

We solve for n_2.

$12 = n_1 + n_2 + 2P_1 - 2n_1$

$n_2 = 12 - 2P_1 + n_1$. Now we rearrange terms.

(8) $12 - 2P_1 = n_2 - n_1$

From (1) we write

$P_1 + P_2 - 2P_1 = n_2 - n_1$

and derive the result:

(9) $P_2 - P_1 = n_2 - n_1$.

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[2] Compare Exx. 1d, 1e and Ex. 3 for an example of the change of arrows among the same points and the values of n_0, n_1, and n_2

(9) shows that the difference between P_1 and P_2 is the difference between the number of arrows that connect points in sector 1 and the number of arrows that connect points in sector 2. Note that if $P_1 = P_2$, then $n_1 = n_2$.

We may reinterpret (9) so that the points are pcs, and redefine P_1 and P_2 as complementary sets. When we interpret the arrows as directed intervals for each directed interval n, the resulting cycles of pcs form the cycles of interval n. Thus n_1 is the number of intervals n *within* the pcset P_1; n_2 = the number of intervals n *within* pcset P_2, and n_0 is the number of intervals n *between* pcsets P_1 and P_2. This means that (9) can be reinterpreted as

(10a) The difference between the cardinalities of complementary pcsets P_1 and P_2 is equal to the difference between number of intervals n within P_1 and the intervals n within P_2.

(10b) If P_1 and P_2 are complementary hexachords, there is no difference between the number of intervals n within P_1 and the number of intervals n within P_2.

Let IF(Z,Z) be the interval function of pcset Z and itself[3] so we can generalize (10a and b).

(11a) Complement Theorem: For complementary pcsets P_1 and P_2, the $IF(P_2,P_2)$ may be derived from $IF(P_1, P_1)$ by adding k to each argument in $IF(P_1,P_1)$, where k is the difference between the cardinalities of P_1 and P_2.

(11b) Hexachord Theorem: If P_1 and P_2 are hexachords, then $IF(P_2,P_2) = IF(P_1,P_1)$.

(11a,b) can be applied to the interval class vectors (ICV) of P_1 and P_2, but the value of the last argument of the vectors has to be divided by 2.

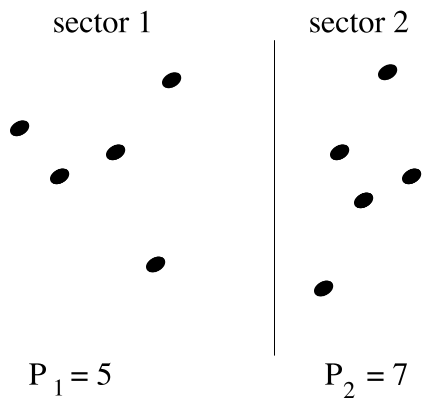
Examples:

Let P_1 = {013467} and P_2 = {2589AB}; k = 0; $IF(P_1,P_1) = IF(P_2,P_2)$ = [632422422423]; $ICV(P_1) = ICV(P_2)$ = [6324222].

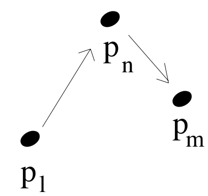
Let P_1 = {0236} and P_2 = {146789AB}; k = 4; $IF(P_1,P_2)$ = [411210201211] and $IF(P_2,P_2)$ = [855654545655]; $ICV(P_1)$ = [4112101] and $ICV(P_2)$ = [8556543].

[3] The interval function is a 12-argument array that lists the intervals between two pcsets, X and Y; it is written IF(X,Y). The nth argument in the array gives the number of directed intervals of size n from X to Y. For example, let X = {024} and Y = {235}. IF(X,Y) = [121201000011]; argument 0 is 1, indicating the one common pc between X and Y; argument 1 is 2, indicates the two interval 1s from 2 in X to 3 in Y and 4 in X to 5 in Y. The ICV(X) and IF(X,X) provide similar information—except in the case of interval 6, as indicated in the text—but ICV(X) gives the number of *interval-classes within* X, where the IF(X,X) gives the number of *directed intervals between* X and a copy of X.

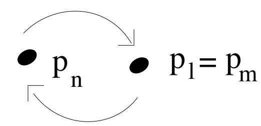
Ex. 1a



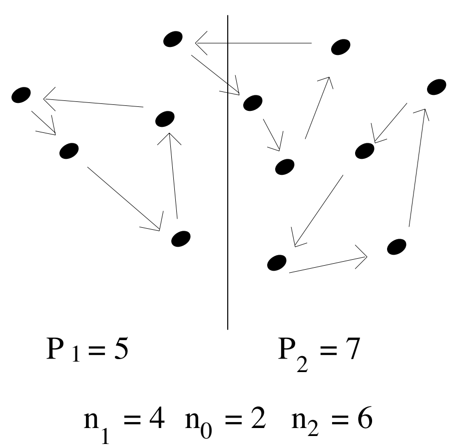
Ex. 1b



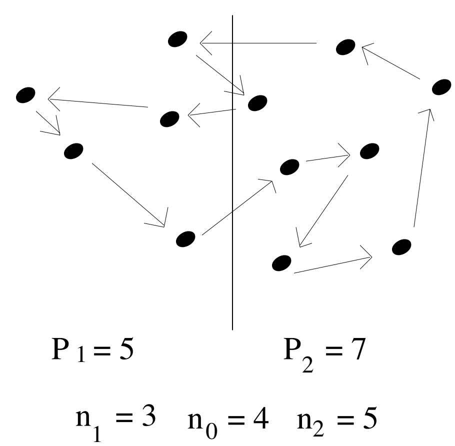
Ex. 1c



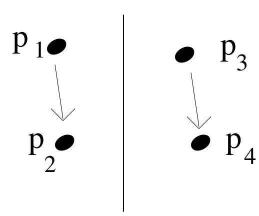
Ex. 1d



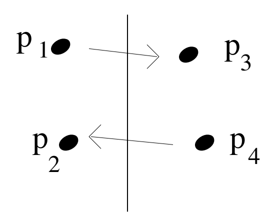
Ex. 1e



Ex. 2a



Ex. 2b



Ex. 3