Mathematics and the Twelve-Tone System
Past, Present, and Future

Robert Morris
Eastman School of Music, University of Rochester
mris@mail.rochester.edu

Outline

1. Rational reconstruction of the twelve-tone system.

2. Major results and trends.

3. Present mathematical knowledge.

4. Future directions.

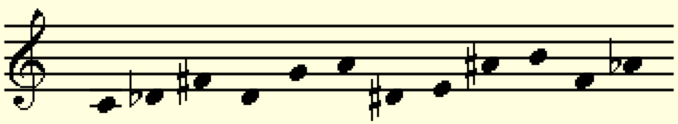
Working definition of the twelve-tone system

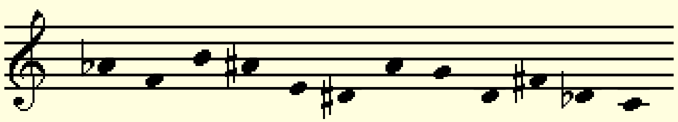
- use of ordered sets of pitch-classes (pcs)
- within the context of the aggregate (the 12 pcs)
- under specified transformations
- (the row is not the nexus of the system)

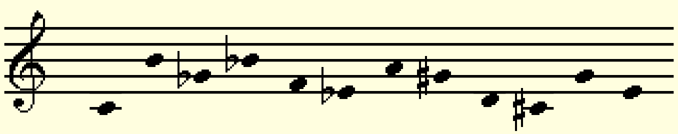
The introduction of math into the twelve-tone system

Ex. 1 The Serial Four-Group.

P	R	I	RI
R	P	RI	I
I	RI	P	R
RI	I	R	P

P:

0 1 6 2 7 9 3 4 A B 5 8

R:

8 5 B A 4 3 9 7 2 6 1 0

I:

0 B 6 A 5 3 9 8 2 1 7 4

RI:

4 7 1 2 8 9 3 5 A 6 B 0

Ex. 2 Twelve-tone invariance among ordered intervals in rows and pairs of rows.

Let the 12-element array P model a row.
The interval between P_a and P_b is written as the function i: $i(P_a,P_b) = P_b - P_a$; $-i(P_a,P_b)$ is the inversion of $i(P_a,P_b)$.

$-i(P_a,P_b) = i(P_b,P_a)$
$i(T_nP_a,T_nP_b) = i(P_a,P_b)$
$i(T_nIP_a,T_nIP_b) = i(P_b,P_a)$

Let the array INT(P) be the interval succession of P; $INT(P) = (i(P_0,P_1), (iP_1,P_2),...i(P_A,P_B))$

$INT(T_nP) = INT(P)$ (T_n preserves the interval succession of P, for all n.)

$INT(T_nIP) = I(INT(P))$ (T_nI inverts the interval succession of P, for all n.)

$INT(RT_nP) = RI(INT(P))$ (RT_n inverts and retrogrades the interval succession of P, for all n.)

$INT(RIT_nP) = R(INT(P))$ (RT_nI retrogrades the interval succession of P, for all n.)

$i(P_a,T_nP_a) = n$ (The interval from pcs in P and T_nP at order position a is n, for all a and n.)

$P_a + T_nIP_a = n$ (The sum of pcs at order number a in P and T_nIP is n, for all a and n.)

$i(P_a,RT_nP_a) = -i(P_{B-a}, RT_nP_{B-a})$ (The interval from pcs in P to RT_nP at order position a is the inversion of the interval from pcs in P to RT_nP at order position B–a, for all a and n.)

$i(P_a,RT_nIP_a) = i(P_{B-a}, RT_nIP_{B-a})$ (The interval from pcs in P to RT_nIP at order position a is the interval between pcs in P to RT_nIP at order position B–a, for all a and n.)

(after Babbitt(1960) and Martino(1961))

Ex. 3 Rows related by shared unordered and ordered sets.

P is the row of Schoenberg's Violin Concerto, op. 36.

Unordered sets shared by related rows.

P:	0 1 6 2 7 9	3 4 A B 5 8
	G	H
RT_5IP:	9 0 6 7 1 2	8 A 3 B 4 5
	G	H

P:	0 1 6	2 7 9	3 4 A	B 5 8
	W	X	Y	Z
T_4IP:	4 3 A	2 9 7	1 0 6	5 B 8
	Y	X	W	Z

Ordered sets shared by related rows.

RT_5IP:

	9	0	6	7	1	2	8	A	3	B	4	5
J:										B		5
K:		0	6			2		A				
L:					1		8		3		4	
M:	9			7								

T_BP:

	B	0	5	1	6	8	2	3	9	A	4	7
J:	B		5									
K:		0			6		2			A		
L:				1		8		3			4	
M:									9			7

Ex. 4 Row succession by complementation and linking.

P is the row of Webern's Variations for Orchestra, Op. 30.

Row succession by complementation (underlined pcs form aggregates from one row to the next)

P: 0 1 4 3 2 5 6 9 8 7 A B / RT_6P: 5 4 1 2 3 0 B 8 9 A 7 6 / T5IP: 5 4 1 2 3 0 B 8 9 A 7 6 / RTBIP: 0 1 4 3 2 5 6 9 8 7 A B / P: 0 1 4 3 2 5 etc.

Row succession by linking

Via T_nRI succession (a twelve-tone invariant)

P: 0 1 4 3 2 5 6 9 8 7 A B / T_8P: 8 9 0 B A 1 2 5 4 3 6 7

RT_9IP: A B 2 1 0 3 4 7 6 5 8 9 /

Via T_5 succession (a special invariance of this row)

P: 0 1 4 3 2 5 6 9 8 7 A B / T_3P: 3 4 7 6 5 8 9 0 B A 1 2

T_5P: 5 6 9 8 7 A B 2 1 0 3 4 / T_8P: 8 9 0 B A 1 2 5 4 3 6 7

T_AP: A B 2 1 0 3 4 7 6 5 8 9

Ex. 5 Berg's Lyric Suite row and other all-interval rows (AIS).

Berg Lyric Suite row:

```
    C: 0 B 7 4 2 9 3 8 A 1 5 6
INT(C): B 8 9 A 7 6 5 2 3 4 1
```

$C = RT_6C$

Wedge row:

```
    D: 0 1 B 2 A 3 9 4 8 5 7 6
INT(D): 1 A 3 8 5 6 7 4 9 2 B
```

$D = RT_6D$

NB: $D = r_6T_9M_5C$

Mallalieu Row:

```
    E: 0 1 4 2 9 5 B 3 8 A 7 6
INT(E): 1 3 A 7 8 6 4 5 2 9 B
```

$E = RT_6E$

rRM_7-invariant Row.

```
    F: 0 5 8 9 3 A 2 4 1 B 7 6
INT(F): 5 3 1 6 7 4 2 9 A 8 B
```

$F = r_4RT_9M_7F$

Krenek's AIS row without special invariance

```
    G: 0 3 A 4 9 B 8 7 5 1 2 6
INT(G): 3 8 6 7 2 9 B A 8 1 4
```

Three stages

- terminological
- conceptual
- methodological

Ex. 6 Common tone and complement theorems.

1. *Transpositional Common Tone Theorem* Let A and B be pcsets. $\#(A \cap T_nB) = MUL(A,B,n)$.

The function MUL(A,B,n) is the multiplicity of i(a,b) = n for all a and b where A and B are pcsets and $a \in A$, $b \in B$.

2. *Inversional Common Tone Theorem*: Let A and B be pcsets. $\#(A \cap T_nIB) = SUM(A,B,n)$.

The function SUM(A,B,n) is number of sums a+b = n, for all a and b where where A and B are pcsets and $a \in A$, $b \in B$.

3. *Complement Theorem*: Let A and B be pcsets. $MUL(A',B',n) = 12-(\#A + \#B) + MUL(A,B,n)$.

is the cardinality operator; #X is the cardinality of X.
´ is the complement operator; A´ is the complement of A.

Important results and trends

- T- and I-matrix
- combinatoriality and arrays
- posets and lattices
- partitions
- special entities
- hierarchy and embedding
- other musical dimensions (time, etc.)
- networks and compositional spaces

Ex. 7a T- and I-matrices for a row and a hexachord/trichord pair.

T-matrix E: $E_{i,j} = P_i + IP_j$

P = 0 1 6 2 7 9 3 4 A B 5 8

	01627934AB58
0	01627934AB58
B	B05168239A47
6	6708139A45B2
A	AB4057128936
5	56B7028934A1
3	3495A067128B
9	9A3B46017825
8	892A35B06714
2	23849B56017A
1	12738A45B069
7	781924AB5603
4	45A6B1782390

I-matrix F: $F_{i,j} = P_i + P_j$

P = 0 1 6 2 7 9 3 4 A B 5 8

	01627934AB58
0	01627934AB58
1	12738A45B069
6	6708139A45B2
2	23849B56017A
7	781924AB5603
9	9A3B46017825
3	3495A067128B
4	45A6B1782390
A	AB4057128936
B	B05168239A47
5	56B7028934A1
8	892A35B06714

T-matrix G: $G_{i,j} = X_i + IY_j$

X={012478}; Y = {348}

	012478
9	9AB145
8	89A034
4	4568B0

I-matrix H: $H_{i,j} = X_i + Y_j$

X={012478}; Y = {348}

	012478
3	3457AB
4	4568B0
8	89A034

Ex. 7b T- and I-matrices generate Lewin's IFUNC.

T-matrix G: $G_{i,j} = X_i + IY_j$

X={012478}; Y = {348}

	0	1	2	4	7	8
9	9	A	B	1	4	5
8	8	9	A	0	3	4
4	4	5	6	8	B	0

IFUNC(X,Y) = [210132102222]

I-matrix H: $H_{i,j} = X_i + Y_j$

X={012478}; Y = {348}

	0	1	2	4	7	8
3	3	4	5	7	A	B
4	4	5	6	8	B	0
8	8	9	A	0	3	4

IFUNC(X,IY) = [200232112122]

$IFUNC_n(X,Y)$ is the number of ns in the T-matrix.
$IFUNC_n(X,IY)$ is the number of ns in the I-matrix.

$IFUNC_n(X,Y) = MUL(X,Y,n)$
$IFUNC_n(X,IY) = SUM(X,Y,n)$

Corollary of Transpositional Common Tone Theorem: $\#(X \cap T_nY) = IFUNC_n(X,Y)$.

Corollary of Inversional Common Tone Theorem: $\#(X \cap T_nIY) = IFUNC_n(X,IY)$.

(# is the cardinality operator; #X is the cardinality of X.)

Ex.7c T-matrix, derived rotational array, transpositional-combination and Tonnetz.

T-mat H: $H_{i,j} = X_i + IX_j$

Rotational Array derived from T-matrix H.
(Diagonals of H become columns of rotational array)

X = 0 A 7 9 2 8*

Columns 0 and 3 are I-invariant; columns 1 and 5 and 2 and 4 are I-related.

T-mat H	Rotational Array
0A7928	0A7928
209B4A	09B4A2
530271	027153
31A05B	05B31A
A85706	06A857
42B160	042B16

*X is the first hexachord of Stravinsky's "A Sermon, A Narrative, and a Prayer."

Transpositional combination of {025} and {289A}
{023456789A} = {025} * {289A}
set-classes 10-3[012345679A] = 3-5[025] * 4-5[0126]

X = {025}; Y = {289A}

	025
2	A03
8	469
9	358
A	247

The Tonnetz is the T-matrix for X = 0369 and Y = 048

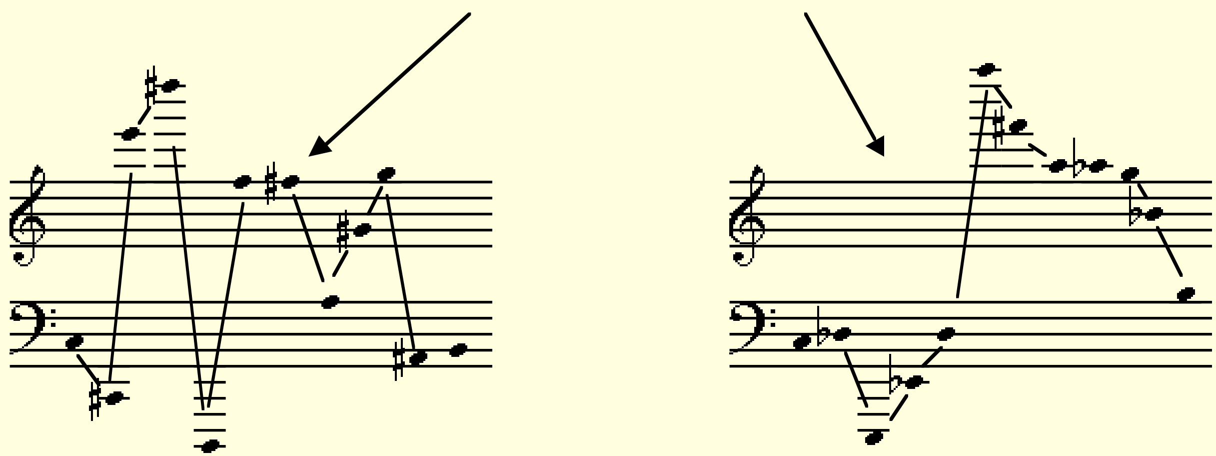
	0369
0	0369
4	47A1
8	8B25

Ex. 7d T-matrix determines permutations between two rows and their interaction to form a determinate contour.

T-mat K: $K_{i,j} = P_i + IP_j$ Submatrices show permutations of P with T_3P and T_5P

```
P = 0143256987AB

   0143256987AB         0143256987AB       0143256987AB
 0|0143256987AB       3|   3        |   5|     5      |
 B|B0321458769A       4|  3         |   6|      5     |
 8|890BA1254367       7|         3  |   9|       5    |
 9|9A10B2365478       6|      3     |   8|        5   |
 A|AB2103476589       5|     3      |   7|         5  |
 7|78BA90143256       8|        3   |   A|          5 |
 6|67A98B032145       9|       3    |   B|           5|
 3|34765890BA12       0|3           |   2|    5       |
 4|458769A10B23       B|           3|   1| 5          |
 5|56987AB21034       A|          3 |   0|5           |
 2|2365478BA901       1| 3          |   3|   5        |
 1|1254367A98B0       2|    3       |   4|  5         |
                        ------------       ------------
```



Ex. 8 Some combinatorial arrays.

(P is the row of Schoenberg, Op. 36.)

Each array column is an aggregate. Each array row is a transformation of P. Below each column is a multiset, identifying the 12-partition of the column.

8a.

P	0 1 6 2 7 9	3 4 A B 5 8
RP	8 5 B A 4 3	9 7 2 6 1 0
	6^2	6^2

8b.

P	0 1 6	2 7 9	3 4 A	B 5 8
RP	8 5 B	A 4 3	9 7 2	6 1 0
T_3IP	3 2 9	1 8 6	0 5 B	4 A 7
RT_3IP	7 A 4	5 B 0	6 8 1	9 2 3
	3^4	3^4	3^4	3^4

8c.

P	0 1 6 2 7	9 3 4 A B 5 8	
T_3P	3 4 9 5 A	0 6	7 1 2 8 B
RT_3P	B 8	2 1 7	6 0 A 5 9 4 3
	$5^2 2$	732	75

8d.

P	0 1 6	2 7 9		3 4 A	B 5 8
T_4IP	4 3 A 2 9 7	1 0 6 5 B 8	(T_BP)	B 0 5 1 6 8	2 3 9 A 4 7
RP	8 5 B	A 4 3		9 7 2	6 1 0
	63^2	63^2		63^2	63^2

8e.

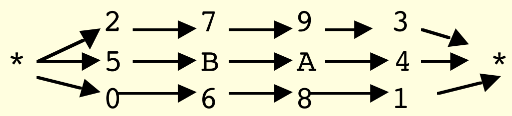
P	0 1 6			2 7 9 3	4 A B 5 8
T_4P	4 5 A	6 B 1	7 8 2 3 9		0
T_9IP	9 8 3	7 2 0	6	5 B A 4	1
RT_3IP	7	A 4 5	B	0 6 8 1	9 2 3
RT_7IP	B 2	8 9 3	4 A 0 5 1		6 7
	$3^3 21$	3^4	$5^2 1^2$	4^3	5321^2

Ex. 9 lattice, poset, and order matrix derived from an array column.

Array column

2	7	9	3
5	B	A	4
0	6	8	1

Lattice derived from Array column



Poset derived from lattice

{ (2,7) (2,9) (2,3) (7,9) (7,3) (9,3) (5,B) (5,A) (5,4) (B,A) (B,4) (A,4) (0,6) (0,8) (0,1) (6,8) (6,1) (8,1) }

Order matrix derived from poset

```
   0123456789AB
0 |010000101000
1 |000000000000
2 |000100010100
3 |000000000000
4 |000000000000
5 |000010000011
6 |010000001000
7 |000100000100
8 |010000000000
9 |000100000000
A |000010000000
B |000010000010
```

Ex. 10 Non-aggregate combinatorial array.

0	245	79	
578			03A
A	037	25	
9		46	18B

Array rows are members of set-class 6-32[024579] (C all-combinatorial hexachord)

Array columns are members of set-class 6-8[023457] (B all-combinatorial hexachord)

Ex. 11 The 77-partitions of the aggregate. (Self-conjugates are in boldface.)

number of parts →

max. length of part ↓	1	2	3	4	5	6	7	8	9	10	11	12
1												1^{12}
2						2^6	$2^5 1^2$	$2^4 1^4$	$2^3 1^6$	$2^2 1^8$	21^{10}	
3				3^4	$3^2 2^3$ $3^3 21$	$32^4 1$ $3^2 2^2 1^2$ $3^3 1^3$	$32^3 1^3$ $3^2 21^4$	$32^2 1^5$ $3^2 1^6$	321^7	31^9		
4			4^3	$43^2 2$ **$4^2 2^2$** $4^2 31$	42^4 $432^2 1$ $43^2 1^2$ $4^2 21^2$	$42^3 1^2$ 4321^3 $4^2 1^4$	$42^2 1^4$ 431^5	421^6	41^8			
5			$5^2 2$ 543	$5^2 1^2$ 5421 $53^2 1$ 532^2	541^3 **5321^2** $52^3 1$	$52^2 1^3$ 531^4	521^5	51^7				
6		6^2	651 642 63^2	641^2 6321 62^3	631^3 $62^2 1^2$	**621^4**	61^6					
7		75	741 732	731^2 $72^2 1$	721^3	71^5						
8		84	831 82^2	821^2	81^4							
9		93	921	91^3								
10		$10\ 2$	$10\ 1^2$									
11		$11\ 1$										
12	12											

Ex. 12a Types of rows.

derived
all combinatorial
all interval

order invariant

$P = r_9RT_4IP$: 0 3 7 B 2 5 9 1 4 6 8 A (Berg, *Violin Concerto*)

all- trichord

Q: 0 1 B 3 8 A 4 9 7 6 2 5 (Babbitt, *Images*)

trichords: 3-1 3-9 3-7 3-3
3-6 3-8 3-2
3-11 3-5 3-4

multiple order function

S: 0 1 B 4 8 5 9 A 7 3 2 6
X: Y: Z:

RT_AIS: 4 8 7 3 0 1 5 2 6 B 9 A
Y: 4 8 5 9 A
Z: 7 3 2 6
X: 0 1 B

S embedded in successions of RT_AIS:

4873015 26B9A487301526B9A4873015 26B9A

set-type saturated (Morris, *Concerto for Piano and Winds*)

T: 0 1 4 7 8 A B 2 5 6 9 3 (0 1 4 7 8)
hexachords: 6-28 6-49
6-49 6-28
6-28 6-49

Ex. 12b Self-deriving row and array. Z: 0 5 1 9 A B 2 7 3 4 6 8

Array:

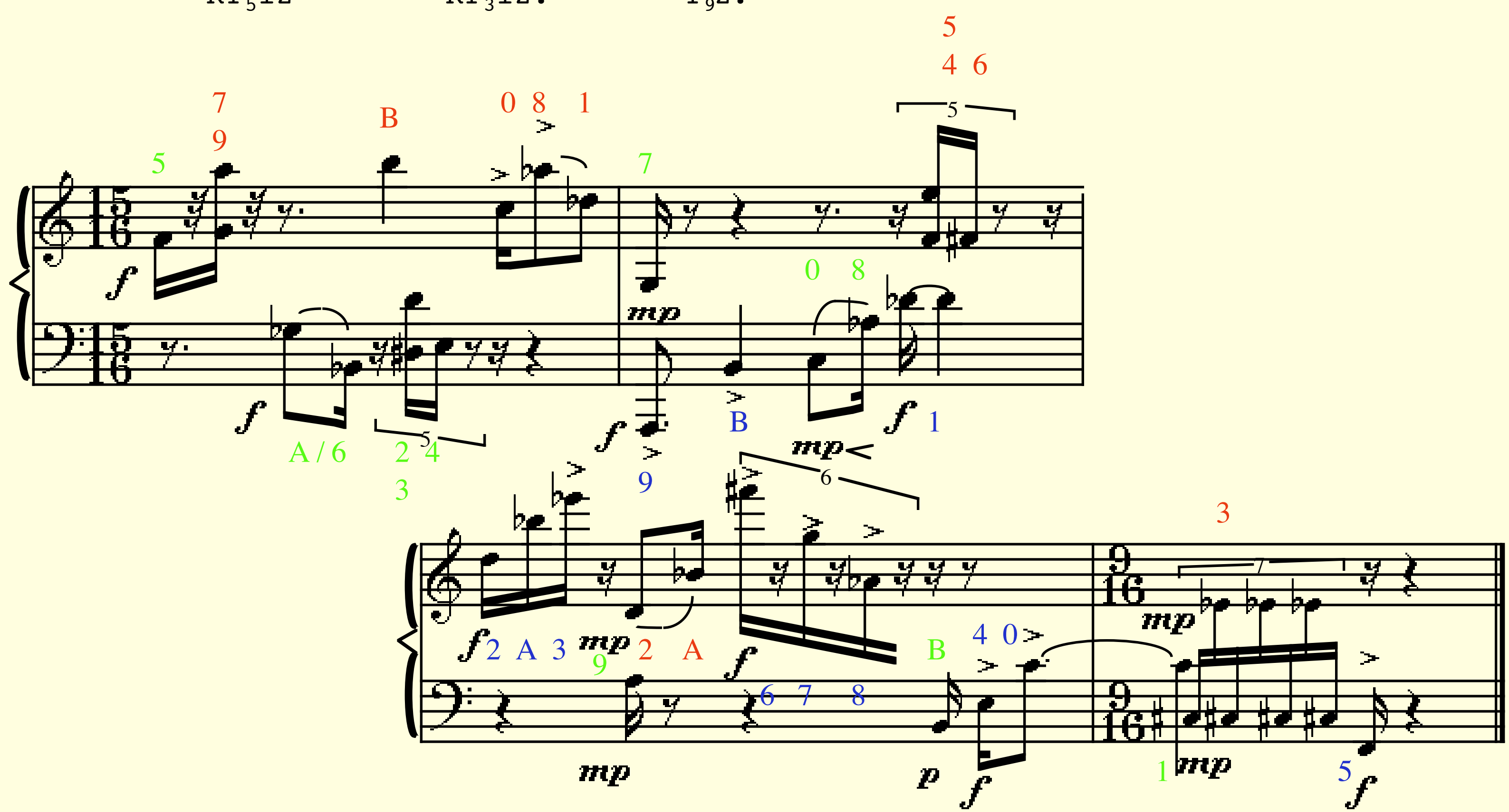
RT_5IZ: | 9B 1 2A3 | 678 40 5 |

RT_3IZ: 79 B 081 | 456 | 2A 3 |

T_5Z: 5 A6 234 | 7 08 | 9 B 1 |

linear rows: 579A6B234081 (RT_5IZ) 79B0814562A3 (RT_3IZ:) 92A678B40135 (T_9Z:)

Piano:



Present mathematical tools and concepts

- affine group acting on Z_{12} or Z
- within context of S_{12}
- context-sensitive groups
- normalization
- also

  semigroups
  fields
  number theory
  combinatorics
  graph theory

Future Research

- other ETS (Z_n)
- similarity of ordered sets
- enumeration (generating functions)
- combinatorial and other arrays
- partitions (Z-relations)
- multisets
- existence proofs