Mathematics and the Twelve-Tone System: Past, Present, and Future

(Reading paper)

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Introduction

Certainly the first major encounter of non-trivial mathematics and non-trivial music was in the conception and development of the twelve-tone system from the 1920s to the present. Although the twelve-tone system was formulated by Arnold Schoenberg, it was Milton Babbitt whose ample but non-professional background in mathematics made it possible for him to identify the links between the music of the Second-Viennese school and a formal treatment of the system

In this paper, I want to do four things.

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First, I will sketch a rational reconstruction of the twelve-tone system as composers and researchers applied mathematical terms, concepts, and tools to the composition and analysis of serial music. Second, I will identify some of the major trends in twelve-tone topics that have led up to the present. Third, I will give a very brief account of our present mathematical knowledge of the system and the state of this research. Fourth, I will suggest some future directions as well as provide some open questions and unproven conjectures.

But before I can start, we need to have a working definition of what the twelve-tone system is, if only to make this paper’s topic manageable.

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Thus I will provisionally define the twelve-tone system as the musical use of ordered sets of pitch-classes in the context of the twelve-pitch-class universe (or aggregate) under specified transformations that preserve intervals or other features of ordered-sets or partitions of the aggregate. Thus the row, while it once was thought to be the nexus of the system, is only one aspect of the whole. Thus an object treated by the twelve-tone system can be a series or cycle of any number of pitch-classes, with or without repetition or duplication, as well as multi-dimensional constructs such as arrays and networks, or sets of unordered sets that partition the aggregate.

SLIDE The introduction of math into twelve-tone music research.

Schoenberg's phrase, "The unity of musical space," while subject to many interpretations, suggests that he was well aware of the symmetries of the system. (Schoenberg, 1975) In theoretic word and compositional deed he understood that there was a singular two-dimensional "space" in which his music lived—that is, the space of pitch and time.

[Ex. 1 Serial four-group with rows.]

Indeed, the basic transformations of the row, Retrograde and Inversion, plus Retrograde-Inversion for closure (and P as the identity) were eventually shown to form a Klein four-group.

That this space is not destroyed or deformed under these operations gives it unity. Yet, from today's standpoint, the details of this symmetry are quite unclear. What kinds of pitch? Pitch, or pitch-class, or merely contour? Is I mirror inversion or pitch-class inversion? Is RI a more complex operation than I or R alone? What about transposition's interaction with the Klein group? And so forth. The lack of clarity, which is actually more equivocal than I've mentioned, fostered misconceptions about the aural reality of the system on one hand and the justification of its application to structuring other so-called parameters of music on the other. Future research would correct this ambiguity, differentiating it into different musical spaces and entities.

Schoenberg nor his students, or even the next generation of European serial composers ever addressed these questions. It was detailed analysis of the music of Schoenberg, Webern, and Berg that led to clarity and rigor. The results of such studies beginning circa 1950 revealed that the first generation of twelve-tone composers had principled reasons for deploying rows in music. First, the system itself was shown to preserve musical properties such as interval and interval-class; Babbitt (1960) called this *twelve-tone invariance*.

[Ex. 2 Twelve-tone invariance among ordered intervals in rows and pairs of rows]

In Example 2, I have distilled identities from Babbitt (1960) and Martino (1961); these identities can easily be derived from the definition of rows, ordered intervals and the twelve-tone operators T_n, I, and R. (This example uses an array to model a row but Babbitt (1960) uses a different concept: A row is a set of unordered pairs, each pair consisting of a pc and its order position in the row).

Second, in addition to twelve-tone invariance, Babbitt and others showed that the rows used by Schoenberg, Berg, and Webern were not chosen capriciously, but would depend on features such as shared ordered and unordered sets. Babbitt (1962) called this *set-structure invariance*.

[Ex. 3 Rows related by shared unordered and ordered sets]

<discuss>

Row succession by complementation or row linking is another aspect of this thinking.

[Ex. 4 Row succession by complementation and linking

<discuss>

These examples demonstrate that musical objects and relations were supported and cross-related from one row to another to build musical continuity, association and form.

Early pre-mathematical research also concerned itself with the relations of the system to tonality. Here are some of the specific questions that arose: was the first pc of a row a kind of tonic; or was a row tonal if it contained tonal material such as triads and seventh chords; did the P and I rows participate in a duality like that of tonic and dominant? In general, tonality was either seen as opposed to the system or both were transcended by a Hegelian sublation into aspects of the same musical and universal laws. But a lack of clarity that conflated reference, quotation, suggestion, analogy, and instantiation made the question impossible to define, much less answer. This obsession with tonality retarded work on the vertical or harmonic combination of rows in counterpoint. Even after the set theories of Howard Hanson (1960), Hubert Howe (1965), and Allen Forte (1964, 1973) had become established, it was not until the 1980s that the problem was generalized to all types of rows and set-classes (Morris, 1983). By this point in time, clarity about the nature of musical systems and their models helped make the tonality issue manageable. Benjamin Boretz's "Meta Variations," published serially in *Perspectives of New Music* from 1969 to 1973 is the seminal work on this topic. Understanding tonality as recursive but invariant among levels made it possible to conceive of the multiple order number function rows (Batstone, 1972) that implement such properties to various degrees. And it was Babbitt who revealed that Schoenberg's later "American" twelve-tone practice was founded on hierarchic principles.

The early research focused on entities. The row was considered the core idea of the system and specific types of rows, such as order-invariant rows or the all-interval-rows (called AIS) were invented (or discovered) and discussed.

For example, the all-interval-row of Berg's Lyric Suite (and also used in other of his works) and its T_6R invariance provides an example.

[Ex. 5 Berg's Lyric Suite row and other AIS.]

<discuss>

Studies of various types of rows continued up until the 1980s, and I will provide examples later.

Questions of enumeration also were raised. How many rows? How many distinct related rows under Transposition, Inversion and/or Retrograde (since some rows are invariant)? Not until the 1960s was it understood that answers to such questions were determined by what transformations one included as canonic—as defining equivalence-classes. (This involves changes in the cyclic index of Burnside's method of counting equivalence classes).

As I have pointed out, it was the lack of adequate formal descriptions and models that limited early work on the twelve-tone system. The introduction of mathematical tools changed all that. By the 1970s it became clear that the system was not only about things, but also about the ways in which these things were changed or kept invariant within the system. In 1978 Daniel Starr explicitly enunciated the entity/transformational distinction that is so familiar to us today. It took some time however before the difference between a binary group and an transformational group was appreciated; or to put it another way, that the set of transformations that formed a group was distinct from the objects it acted on; and that these objects might be not only pitch-classes, but sets, arrays, networks, etc., which in turn might suggest a variety of types of transformation groups. (Lewin 1978, Morris, 1978.) This widened the scope of twelve-tone theory to encompass non-twelve-tone things such as tonal chords, scales, and the like.

SLIDE

The intervention of mathematical tools occurred in three stages—the terminological, conceptual, and methodological. First was the use of mathematical terminology and symbols including the use of numbers to identify pitch-classes, order numbers, and transpositional levels. Variable names (with subscripts) such as S_n or P_n, I_n, R_n, and RI_n were used to name rows. However, this practice conflated the difference between a label denoting an entity versus a transformation.

A second stage was the use of mathematical and logical concepts such as equivalence and relation, and the use of mathematical terms borrowed from real math or computer science such as “invariance” or “function.” Sometimes, strange terminology from the mathematical point of view resulted: such as the names “set-class” or “interval-vector”; or using the term “complement” to mean “inverse.” But at least these ideas and functions were more or less contextually well defined. At this stage, concepts were generally used to describe the properties of musical entities. Perhaps the most important insight was Babbitt’s claim that the twelve-tone system was inherently permutational rather than combinational. (Babbitt, 1960) While this assertion is perhaps too categorical,[1] Babbitt opened the door to the use of group theory in musical research. Researchers also adopted the language of set theory to describe musical properties and relations among sets of musical things. Nevertheless, confusion remained because the same terms were used for different kinds of things. For instance, in the 1970s the term “set” meant row at Princeton and unordered-pc-set at Yale. Moreover, technical labels did not address all the important differences. The distinction between interval and interval-class was not explicitly defined; later, the interval-class would finally be understood as the “distance” between pitch-classes or pitches, while the term “interval” would define a directed distance between two pitch-classes or a transformation of one to another. Sets and set-classes were still not adequately distinguished in the literature until around 1975, even after the publication of Allen Forte’s important book (1973), which does not explicitly make the distinction.

[Ex. 6 Common tone and complement theorems]

[1] Tonality and theories of chords involve permutation and aspects of the twelve-tone system involve unordered sets.

The third stage involved the use of mathematical reasoning in music theory. At first this reasoning would be alluded to, or presented in words, or in symbols in ad hoc ways. Sometimes this work was done behind the scenes, as in the proof of the complement theorem, which was asserted in the late 1950s but not explicitly proven in the literature until the 1980s.[2]

But it didn't take long before there were ways to do something like professional mathematics in the body of a music theory paper. This led to some consensus about the nature of the terminology and formalisms used in music theory today—but sometimes these do not correspond one-to-one with mathematical treatment. With the use real mathematics in music theory, theorists realized that there are branches of mathematics that could be applied to their problems; up to then many theorists constructed the mathematics needed from the ground up.

The transition from stage two to three was aided by the use of computers to model and/or enumerate aspects of the twelve-tone system. Starting circa1970, many graduate programs introduced faculty and students to computer programming via seminars and courses. The result was an appreciation of the need for correct and apt formalization of music theoretic concepts and reasoning. This paved the way for researchers to go directly into the math that underlay the design and implementation of the computer programs. Moreover, the output of programs posed new puzzles. What was the stucture underlying the output data?

These three stages actually overlapped in the literature depending on the mathematical sophistication of both authors and readers. Some mathematical treatments of serial topics remained virtually unread until music theory as a whole caught up.

For instance, Walter O'Connell (1963) wrote a mathematically interesting and prescient article in *Die Reihe* 8; however, theorists and composers have generally overlooked it

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[2] The theorem was enunciated as early as Hanson (1960), and sketches for a proof were given in Regener (1974) and Starr (1978). An elegant proof appears in Lewin 1987.

even though it is the first published account of the multiplicative pitch-class operations, the order-number/pitch number exchange operator, and networks of pitch-classes and transformations in multiple dimensions. Sometimes such work was not even published or, if published, criticized as irrelevant to music study—as unwanted applied mathematics. The prime example involves the classical papers by David Lewin on the Interval Function. Lewin's sketch of the mathematical derivation of the function via Fourier analysis, published in JMT in 1959 and 1960 was not appreciated and developed until recently by young theorists such as Ian Quinn (2006).
SLIDE Important results and trends

As you can see, the slide shows some of the important results in the past 50 years.

Perhaps the most important development in twelve-tone theory was the invention of invariance matrices of Bo Alphonce at Yale in 1974.

[Ex. 7a T- and I-matrices for a row and a hexachord/trichord pair]

Here T- and I-matrices are shown to display properties of pairs of ordered or unordered sets.
<discuss>

In addition, Alphonce used them to analyze one passage of music in terms of another. Since the row-table (probably invented by Babbitt in the 1950s) is a special case of the T-matrix, the complex of rows was shown to be related to its generating row in ways supplementing those already formalized by earlier research such as the common-tone and hexachord theorem.

[Ex. 7b T- and I-matrices generate Lewin's I-func.]

From one point of view, the T-matrix is a complete list of the directed intervals between the entities that generate it.

<discuss>

[Ex. 7c T-matrix, derived rotational array, transpositional-combination and Tonnetz.]

It has many other functions and uses, such as spelling out the verticals in Stravinsky's rotational arrays, since those array's columns are the diagonals of the T-matrix. The matrix performs transpositional combination or Boulez's multiplication. Moreover, the *Tonnetz* is a T-matrix.

[Ex. 7d T-matrix displays permutations between two rows and their interaction to form a determinate contour.]

As you can see, the T-matrix can show permutation matrices that determine contours among row presentations.

In my 1987 book, invariance matrices underlie and unify many different aspects of serial theory including the relations of sets of transformations and mathematical groups. This is because a T-matrix is a group table or a part thereof.

I've already mentioned Babbitt's important articles on serial music. His earliest work, including the first work on combinatoriality—that is, aggregate preservation among contrapuntal combinations of rows, as documented in his 1947 Princeton dissertation is quite an achievement, for Babbitt was able to make progress without the explicit distinction of pitch and pitch-class, operator and entity, and set and set-class, and without any explicit invocation of group theory.) Babbitt (1961, 1973), Donald Martino (1961), Starr and myself (Starr and Morris, 1977-78) continued to develop the theory of combinatoriality.

[Ex. 8 Some combinatorial arrays]
<discuss>

It was established that while small combinatorial arrays (as shown in the example) depended on the properties on the generating row, larger and more elaborate arrays depended on more global principles. Consequently, the emphasis shifted from the row to the array so that the array might be considered the more basic musical unit. (Winham, 1970, Morris, 1983, 1987) This was inherent in Babbitt's serial music, which, while unnoticed for quite a time in the literature, had been composed from pairs of combinatorial rows rather than rows alone—that is, from two-part arrays. Thus various types of posets of the aggregate and their possible realization as rows became the focus of this research. Lewin (1976) and Starr (1984) were the first to specify and formalize the use of posets in twelve-tone theory.

[Ex. 9 lattice, poset, and order-matrix derived from an array column]
<discuss>

Eventually the array concept became detached from aggregates and rows so that it could model the preservation of harmonic relations among simultaneous linear presentations of any kinds of pitch or pitch-class entities. (Morris, 1983, 1987, 1995a)

[Ex. 10 Non-aggregate combinatorial array]
<discuss>

Such non-aggregate combinatoriality was useful in formalizing and extending aspects of the music of Carter and others. The topic extends into set-type saturated rows, two-partition graphs and the complement-union property. (Morris 1985, 1987)

[Ex. 11 The 77-partitions of the aggregate]

Research on partitions of the aggregate form a related trend to combinatoriality. Babbitt was the first composer to use all 77 partitions of the number 12 in his music by inventing the all-partition array. (Babbitt 1961, 1973) The earliest emphasis on partitions is that of

Hauer, whose tropes are collections of 6/6 partitions grouped by transposition. Martino's article of 1961 is an early development of the partitions of the aggregate followed more than 25 years later by Andrew Mead (1988), Harald Fripertinger (1992), Brian Alegant (1993), and Alegant and Lofthouse (2002).

As I said, studies of kinds of rows have led to generalities beyond rows. The next example provides a brief survey of some of these special rows. These types are not mutually exclusive so that a row might reside in all of these categories.

[Ex. 12a Types of rows]

(The example does not show derived, all-combinatorial, and all-interval rows because I've already given examples of these types.)
<discuss>

Another kind of row, difficult to determine by inspection, permits self-deriving arrays.

[Ex. 12b Self-deriving arrays]
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These examples point out that research on these rows by Batstone (1972), Scotto (1995), (Morris, 1976, 19771985), (Starr 1984), and Kowalski (1985) reflected new orientations to the use and function of the twelve-tone system, which developed, in turn, into considerations of various kinds of saturation in addition to aggregate completion, the embedding of one musical thing in itself or another, the preservation of properties among like entities such as ordering, transformations, and set-structure. These topics are grounded in the cycles of transformations considered as permutations and the orbits of the permutation groups. These questions of preservation often hinge on whether pairs of transformations commute, and if their orbits and cycles are invariant under interval preserving transformations. Mead's (1988) elaboration on the pc/order number isomorphism introduced by Babbitt and O'Connell is another signal contribution to this topic for it allows any subset of an ordered pc entity to be characterized as batches of pcs at batches of order numbers or vice versa; in this way, all partitions of the aggregate are available in each and every row and the difference between rows is based on the distributions of these partitions over the class of all rows.

The development of ways to extend adequately the relationships among pitch-classes to time and other musical dimensions, was an unsolved problem until the advent of Stockhausen's article "...how time passes..." (1959) and Babbitt's (1962) time-point system. Such elaborations were further developed by Rahn (1972), Morris (1987) and especially David Lewin (1987), who constructed non-commutative temporal GISs that did not preserve simultaneity, succession or duration.

Another line of research concerns the construction of networks of pitches or other musical entities connected by succession, intersection or transformation. Perle's (1977) elaboration of his cyclic sets together with Lansky's (1973) formalization via matrix algebra, and the further generalizations to K-nets (Lewin, 1990) represents one strand in network theory. Another strand is the use of networks of protocol pairs to create poset lattices for generalizing order relationships in serial music (Lewin, 1976; Starr 1984). Yet another strand begins with similarity graphs among pcsets and set-classes (Morris, 1980), two-partition graphs Morris, 1987), transformation networks (Lewin, 1987) and some types of compositional spaces (Morris, 1995a). (John Rahn's recent article in the *Journal of Mathematics and Music* provides some mathematical distinctions and nuances among networks.)

In the interest of time and space, I've left out a great deal of important research including the application pc theory to musical contour and time.

SLIDE Present mathematical tools

Today, the nature of the twelve-tone system is well understood. In a few words, the field is supported by an application of mathematical group theory, where various kinds of groups act on pcs, sets, arrays, etc. The most important group is the affine group including the T_n, and M_m operations acting on Z12 or simply Z. Other subgroups of the background group S_{12} have been used to relate musical entities; these fall into two categories; the so-called context sensitive groups some of which are simply-transitive, and groups that are normalized by operations in the affine group. Other branches of math having strong connections with group theory such as semi-groups and fields, number theory, combinatorial analysis and graph theory are often implicated in twelve-tone research.

What is more, when it became obvious that serial theory was actually an application of group theory, research shifted over from modeling serial composition and analysis to other aspects of music that involved symmetry. David's Lewin's (1987) work on general interval systems (GIS) and transformation networks represents this change of orientation. Thus, the development of the twelve-tone system has been so extended and ramified that there is no longer a need to distinguish this line of work from other mathematically informed branches of theory. Neo-Reimannian, scale theory, networks, and compositional spaces, unify and interconnect music theory in hitherto unexpected ways. Thus the distinction between tonal and atonal may no longer very meaningful; rather, distinctions between types and styles of music are much more context-sensitive and nuanced thanks to the influence of mathematics.

SLIDE Future Research with outline.

While the twelve-tone system is no longer isolated from other aspects of music theory, there are many research projects that can be identified to carry on previous work,

One obvious direction is to ask what happens when we change the "12" in twelve-tone system? Carlton Gamer (1967a and b) was one of the first theorists to raise such issues. He showed that equal tempered systems of other moduli not only have different structures, they allow different types of combinatorial entities to be built within them. Another aspect that individuates mod-n systems is that its (multiplicative) units need not be their own inverses as they are in the twelve-tone system. Moreover, when n is a prime, all integers mod n are units. Jumping out of any modular system into the pitch-space, there are other ways of conceptualizing and hearing pitch relations, as in spectral composition.

Let me list a few more specific research issues.

What are the ranges for models of similarity between and among ordered sets (including rows)? A few models have been introduced: order-inversions (Babbitt, 1961), BIPs (Forte, 1973), and the correlation coefficient (Morris, 1987). At the time of this writing, Tuukka Ilomaki is working a dissertation on row similarity.

Generating functions and algorithms have been useful in enumerating the number of entities or equivalence classes such as rows, set-classes, partition-classes, and the like. Are there mathematical ways of generating entities of certain types such as all-interval series or multiple order function rows? Some preliminary results are found in Fripertinger (1992). Babbitt has pointed out that the famous multiple order function Mallalieu row can be generated by the enumeration of imprimative roots. Can most or all multiple order function rows be similarly generated? Caleb Morgan has been working on this question and will soon publish the results.

There is much work to be done on the generation of combinatorial and other arrays. For instance, it is an unproven conjecture that any row can generate a twelve-row, all 77-partition array, but only special rows can generate a 4-row, all 34-partition array. However, in the later case, even the necessary criteria are not known. Bazelow and Brickle carried out an initial probe into this problem in 1976. A host of other similar problems surround the creation and transformation of arrays.

In twelve-tone partition theory, the Z-relation is generally understood, but what about in systems of other moduli? David Lewin showed that there were Z-triples in the 16-tone system. (Lewin, 1982). Does the Z-phenomenon have one root cause or many?

Multisets are of use for modeling doubling and repetition in voice-leading and weighted arrays. Even the most basic questions of enumeration and transformation of multi-pcsets have yet to be investigated.

Existence proofs have been lacking to explain why—for instance—there are no all-interval rows that are also all-trichordal.[3] Another open question is if there exist 50-pc rings that imbricate an instance of each of the 50 hexachordal set-classes?

Conclusion

The introduction of math into music theoretic research has had a number of important consequences. At first, the work simply became more rigorous and pointed in the questions that it could be ask and in the generality of the answers. On one hand, this led to the identification of different types of twelve-tone music and the models for each type within the twelve-tone system. On the other hand, group theory eventually unified what seemed to be different aspects of music so that the twelve-tone system could no longer completely be conceptually differentiated from tonality, modality, and even aspects of non-Western music. I say, “not completely differentiated,” for there are other mathematical bases for music besides group theory. For instance, Schenkerian tonal theory is not modeled by groups of transformations; here we have tree structures.

In any case, while there can be no doubt that the way we regard music has been transfigured by the use of math in music theory, the music we study remains or remains to be written.

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[3] Here we mean all-trichordal in Babbitt sense of the term: a row that imbricates an instance of each of ten different trichordal set-classes, leaving out [036] and [048].