

Mathematics and the Twelve-Tone System: Past, Present, and Future

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The first major encounter of non-trivial mathematics and non-trivial music was in the conception and development of the twelve-tone system from the 1920s to the present. Although the twelve-tone system was formulated by Arnold Schoenberg, it was Milton Babbitt whose ample but non-professional background in mathematics made it possible for him to identify the links between the music of the second-Viennese school and a formal treatment of the system. This paper sketches a rational reconstruction of the twelve-tone system as composers and researchers applied mathematical terms, concepts, and tools to the composition and analysis of serial music. It also identifies some of the major trends in twelve-tone topics that have led up to the present, provides a brief account of our present mathematical knowledge of the system, and suggests some future directions for research, including open questions and unproven conjectures.

What is the twelve-tone system? For this purposes of this paper it is described as the musical use of ordered sets of pitch-classes in the context of the twelve-pitch-class universe (or aggregate) under specified transformations that preserve intervals or other features of ordered-sets or partitions of the aggregate. Thus the objects treated by the twelve-tone system are not only rows, but any series or cycle of any number of pitch-classes, with or without repetition or duplication, or multi-dimensional constructs such as arrays and networks.

The earliest research into the nature of the system is captured by Schoenberg's phrase, "The unity of musical space," Undoubtedly, Schoenberg understood this space as the combination of pitch and time. But aside from the basic transformations of the row, R and I, plus RI for closure (and P as the identity), the details of this space are quite vague. Similar lapses of clarity fostered misconceptions about the aural reality of the system on one hand, and the justification of its application to structuring other so-called parameters of music on the other. Future research would correct this ambiguity, differentiating it into different musical spaces and entities. First, the system itself was shown to preserve musical properties such as interval and interval-class (Babbitt, 1960); Second, Babbitt (1962) showed that rows were not chosen by Schoenberg and his students capriciously, but would depend on features such as shared ordered and unordered sets.

Early pre-mathematical research also concerned itself with the relations of the system to tonality. But once again a lack of clarity that conflated reference, quotation, suggestion, analogy, and instantiation made the question impossible to define, much less answer. This obsession with tonality retarded work on the vertical or harmonic combination of rows in counterpoint. It took until the 1970s for research on the nature of musical systems and their models helped make the tonality issue manageable. Understanding tonality as recursive but level invariant made it possible to conceive of the multiple order number function rows (Batstone, 1972).

Early twelve-tone research focused on entities. The row was considered the core idea of the system and specific types of rows, such as order-invariant rows or the all-interval-rows were invented (or discovered) and discussed. The trope system of Hauer (1925), contemporaneous with the invention of the twelve-tone system, concerned partitions. Questions of enumeration also were raised, but not until the 1960s was it realized that the answers were determined by what transformations one included as canonic—as defining equivalence-classes.

By the 1970s it became clear that the system was not only about things, but also about the ways in which these things were changed or kept invariant within the system. In 1978 Daniel Starr enunciated the entity/transformational distinction that is so familiar to us today. This eventually widened the scope of twelve-tone theory to encompass non-twelve-tone things such as tonal chords, scales, and the like.

The intervention of mathematical tools occurred in three stages. First was the use of mathematical terminology and symbols including the use of numbers to identify pitch-classes, order numbers, and transpositional levels. A second stage was the use of mathematical and logical concepts such as equivalence and relation, and the use of mathematical terms borrowed from real math or computer science such as “invariance” or “function.” Perhaps the most important insight was Babbitt’s assertion that the twelve-tone system was inherently permutational rather than combinational. (Babbitt, 1960); this opened the door to the use of group theory in musical research. Researchers also adopted the language of set theory to describe musical properties and relations among sets of musical things.

The third stage involved the use of mathematical reasoning in music theory. Sometimes this work was done behind the scenes, as in the proof of the complement theorem, which was asserted in the late 1950s but not explicitly proven in the literature until the 1980s.¹ But it didn’t take long before there were ways to do something like professional mathematics in the body of a music theory paper. Sometimes this work was original or unique from the mathematical point of view. The transition from stage two to three was also aided by the use of computers to model and/or enumerate aspects of the twelve-tone system.

These three stages—the terminological, conceptual, and methodological—actually overlapped in the literature depending on the mathematical sophistication of the author and reader. Thus some mathematical treatments of serial topics remained virtually unread until theory as a whole caught up.

Perhaps the most important development in twelve-tone theory was the invention of invariance matrices of Bo Alphonse at Yale in 1974. Here T- and I-matrices were shown to display the properties of pairs of ordered or unordered sets. In my 1987 book, invariance matrices underlie and unify many different aspects of serial theory including

¹ The theorem was enunciated as early as Hanson (1960), and sketches for a proof were given in Regener (1974) and Starr (1978). An elegant proof appears in Lewin 1987.

the relations of sets of transformations and mathematical groups. This is because a T-matrix is a group table or a part thereof.

Babbitt's important articles on serial music were to develop the compositional practices of Schoenberg and Webern into a system of combinatoriality or aggregate preservation among contrapuntal combinations of rows. Donald Martino (1961), Daniel Starr and myself (Starr and Morris, 1977-78) continued to develop the theory of combinatoriality. Consequently, the emphasis shifted from the row to the array so that the array might be considered the more basic musical unit. (Winham, 1970, Morris, 1983, 1987) But arrays are also collections of posets. Lewin (1976) and Starr (1984) were the first to specify and formalized the use of posets in twelve-tone theory. Eventually the array concept became detached from aggregates and such non-aggregate combinatoriality (Morris, 1983) was useful in formalizing and extending aspects of the music of Carter and others.

Research on partitions of the aggregate form a related trend to combinatoriality. Babbitt was the first composer to use all 77 partitions of the number 12 in his music by inventing the all-partition array. (Babbitt 1961, 1973) The earliest emphasis on partitions is that of Hower, whose tropes are collections of 6/6 partitions grouped into partitions related by transposition. Martino's article of 1961 is an early development of the partitions of the aggregate followed more than 25 years later by Andrew Mead (1988) and Brian Alegant (1993) and Alegant and Lofthouse (2002).

Studies of kinds of rows have led to generalities beyond rows. The research on derived, all combinatorial rows, all interval rows, order invariant rows, all-trichord rows, multiple order function rows (Batstone 1972; Morris 1976, 1977; Mead 1988, 1989; Scotto, 1995), set-type saturated rows (Morris 1985) and self-deriving array rows (Starr 1984; Kowalski, 1985) has developed into considerations of various kinds of saturation in addition to aggregate completion, the embedding of one musical thing in itself or another, the preservation of properties among like entities such as ordering, transformations, and set-structure. Mead's (1988) elaboration on the pc/order number isomorphism introduced by Babbitt (1960) and O'Connell (1958) further elaborated the properties of these special rows.

Other developments included ways to extend the relationships among pitch-classes to time and other musical dimensions (Stockhausen, 1959 and Babbitt, 1962) and the construction of networks of pitches or other musical entities connected by succession, intersection or transformation. two-partition graphs (Morris, 1987), transformation networks (Lewin, 1987) and some types of compositional spaces (Morris, 1995a).

Today, the field is supported by an application of mathematical group theory, where various kinds of groups act on pcs, sets, arrays, etc. The most important group is the affine group including the T_n , and M_m operations either in Z_{12} or simply Z . Other subgroups of the background group S_{12} have been used to relate musical entities; these fall into two categories; the so-called context sensitive groups some of which are simply-transitive, and groups that are normalized by operations in the affine group. Other branches of math having strong connections with group theory such as semi-groups and

fields, number theory, combinatorial analysis and graph theory are often implicated in twelve-tone research.

When it became obvious that serial theory was actually an application of group theory, research shifted over from modeling serial composition and analysis to other aspects of music that involved symmetry. David's Lewin's (1987) work on general interval systems (GIS) and transformation networks represents this change of orientation. Thus, the development of the twelve-tone system has been so extended and ramified that there is no longer a need to distinguish this line of work from other mathematically informed branches of theory. As a result, distinctions between types and styles of music are much more context-sensitive and nuanced thanks to the influence of mathematics.

While the twelve-tone system is no longer isolated from other aspects of music theory such as models for tonality, there are many research projects that can be identified to carry on previous work. For instance: what happens when we change the "12" in twelve-tone system to n ? How does this affect the Z-relation or the possible properties of entities? What are some models of similarity between and among ordered sets (including rows). Are there ways of generating entities of certain entities such as all-interval series or multiple order function rows? How do the structures of rows influence the generation of combinatorial and other arrays? What properties of multisets are of use in modeling doubling and repetition in voice-leading and weighted arrays? What of existence proofs to show that certain complex musical entities are possible or not?

The introduction of math into music theoretic research has had a number of important consequences. At first, the work simply became more rigorous and pointed in the questions that it could be ask and in the generality of the answers. On one hand, this led to the identification of different types of serial music and models for each type within the twelve-tone system. On the other hand, group theory eventually unified what seemed to be different aspects of music so that the twelve-tone system is no longer completely conceptually differentiated from tonality, modality, and even aspects of non-Western music.