

Rhythmic Symbols, Duration, Tempo, Metric Modulation

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1) Rhythmic symbols.


Unlike the standard notation for pitch,¹ the standard rhythmic symbols are variables. This is because the duration of rhythmic symbols such as eighth notes changes depending on the tempo.


2) Durations.


If we know the duration of one rhythmic symbol, we know the duration of all the rest. For instance, if a quarter note lasts 2 seconds, an eighth notes last one second and a sixteenth-sextuple note lasts a third of a second.


3a) Calculation.


For the purpose of calculation we shall use the following symbols:

W = 

H = 

Q = 

E = 

S = 

T = (thirty-second note)

¹ When we write F# on the first space of the treble clef, the symbol stands for one pitch only (given A, a minor third higher, is set to 440 Hz). The use of transposition for instruments like the horn or string bass is an exception in one sense, but within the parts written for these instruments the pitches written on the staff are invariable. Thus, to within a given instrumental part, a pitch symbol on the staff is a constant, not a variable.

We understand from the standard relations in rhythmic notation that

$$2T = S; \quad 2S = E; \quad 2E = Q; \quad 2Q = H; \quad 2H = W$$


and

$$\frac{W}{2} = H; \quad \frac{H}{2} = Q; \quad \frac{Q}{2} = E; \quad \frac{E}{2} = S; \quad \frac{S}{2} = T.$$

3b) Triplets.

Dividing rhythmic symbols by n, for an n-tuplet note, represents triplets.

Therefore, $\frac{Q}{3} = \text{triple eighth note}$ and $\frac{E}{5} = \text{quintuplet eighth note}$

More complex triplets can be defined just as easily; for instance, $\frac{E+S}{7}$ is the value of a septuplet-note, seven of which equal a .

We can define triplet notes values in terms of W.

$$\frac{Q}{3} = \frac{W}{12} \text{ (since } 3 \frac{Q}{3} = Q \text{ and } 4Q = W \text{ so } 12 \frac{Q}{3} = W \text{ and } \frac{Q}{3} = \frac{W}{12} \text{)}$$

or $\frac{Q}{5} = \frac{W}{20}$.

3c) Special time signatures.

Some composers have used special time signatures based on these identities; for instance, the time signature $\frac{5}{20}$ indicates there are five sixteenth note quintuplets in a measure (which is the same duration of a measure of $\frac{1}{4}$ time). $\frac{12}{14}$ time indicates that each measure has 12 eighth note septuplets; like this:



The composer might even be more explicit, specifying the 12 “beats” to be grouped into four equal parts:



These types of time signature can be used instead of metric modulations.

3d) Adding durations.

As we have already implied, to add together n identical rhythmic values, we simply multiply by n. So, $E + E + E = 3E$

When we have to add together different rhythmic values, we have to obey the rules of adding fractions taught in high school.

As an example. let us add together the three rhythmic symbols: $\overset{3}{\text{♩}}$ ♩ .

We have $\frac{Q}{3} + H + E$

We transform each value so it is in relation to Q.

$$\frac{Q}{3} + 2Q + \frac{Q}{2}$$

We perform the addition of fractions.

$$\left(\frac{2}{2}\right)\frac{Q}{3} + \left(\frac{6}{6}\right)2Q + \left(\frac{3}{3}\right)\frac{Q}{2}$$

$$\frac{2}{6}Q + \frac{12}{6}Q + \frac{3}{6}Q$$

$$\frac{2Q + 12Q + 3Q}{6} = \frac{17}{6}Q = 2\frac{5}{6}Q$$

Thus, the result is equal to a half note followed by 5 sextuplet notes. If ♩ = 1 second, the result is equal to 2.83333... seconds.

3e) Dotted values.

To prevent ambiguities (confusing mensuration dots, periods and decimal points), we will not use dotted values in our calculations. So ♩. = 3E.

However, we can formalize dotted values so that a rhythmic symbol S followed by n dots is equivalent to $\sum_{k=0, n-1} \frac{S}{2^k}$

$$\text{Therefore, } \text{♩}\dots = \sum_{k=0, 2} \frac{H}{2^k} = \frac{H}{2^0} + \frac{H}{2^1} + \frac{H}{2^2} = \frac{H}{1} + \frac{H}{2} + \frac{H}{4} = H + Q + E = 7E.$$

4a) Metronome markings.

“B MM”, where B is a number means that a metronome produces B (even) clicks in a minute. So 67 MM means the metronome produces 67 clicks in a minute. 120 MM means there are 120 clicks in a minute or 2 clicks a second.

Now the duration between clicks is equal to $\frac{1}{B}$ minutes, or $\frac{60}{B}$ seconds.² If we use d for the duration between clicks, we have

$d = \frac{60}{B}$ seconds. This gives us the number of seconds per duration.

We can also write

$\frac{1}{d} = \frac{B}{60}$ seconds. This gives us the number of durations per second.

Note that d (#seconds per duration) and $\frac{1}{d}$ (#durations per second) are reciprocals, so that we need only invert the fraction equaling d to get the fraction equaling $\frac{1}{d}$

Example:

Let us set a metronome to 80. So B = 80.

The duration of each click is $\frac{3}{4}$ seconds since: $d = \frac{60}{80} = \frac{3}{4}$

There are $1\frac{1}{3}$ clicks per second because: $\frac{1}{d} = \frac{80}{60} = \frac{4}{3} = 1\frac{1}{3}$

(As indicated above, $\frac{3}{4}$ and $\frac{4}{3}$ are reciprocals.)

² Note that B and d are inversely proportional. $d \times B = 1$ minute or 60 seconds. We can say B is the frequency of the clicks and d is the wavelength.

4b) Tempos.

Tempos are given by the marking “N = B MM”, where N is a rhythmic value and B is a number. For instance,

$$\downarrow = 69 \text{ MM} \quad \text{or} \quad \downarrow = 132 \text{ MM}.$$

Since $d = \frac{60}{B}$ seconds, the duration of the rhythmic value is d. So in the first case above,

$$Q = \frac{60}{69} = \frac{20}{23} \text{ seconds}; \text{ and in the second case, } 3Q = \frac{60}{132} = \frac{5}{11} \text{ seconds}.$$

4c) Duration of pieces.

We can calculate the duration of a piece of music from the counts of measures and the tempo marking.

If we have 32 measure piece in 12/8 time at $\downarrow = 90 \text{ MM}$, $3E = \frac{60}{90} = \frac{2}{3}$ seconds per \downarrow ;

there are 4 in a measure and 32 measures. Thus the duration of the piece is $\frac{2}{3} \times 4 \times 32$ or $85\frac{1}{3}$ seconds.

Another example. The piece starts out in $\frac{5}{8}$ time with a measure equaling 108 MM. The piece is 7 measures long: 4 measures of $\frac{5}{8}$, 1 measure of $\frac{3}{8}$, 1 measure of $\frac{11}{8}$, and 2 measures of $\frac{5}{8}$. The duration of the piece is therefore equal to

$$4 \times 5E + 3E + 11E + 2 \times 5E = 44E. \text{ Now } 5E = \frac{108}{60} = \frac{9}{5} \text{ seconds and } E =$$

$$\frac{9}{5} \div 5 = \frac{9}{5 \times 5} = \frac{9}{25} \text{ seconds}.$$

$$\text{The duration of the piece is } 44E = 44 \times \frac{9}{25} = 15.84 \text{ seconds}.$$

5a) Metric modulations.

Metric modulations may be written in two ways. At some juncture in a piece, A) the metronome indication changes, or B) some rhythmic value before the juncture is set to equal a different rhythmic value after the juncture. Here is an example of each type of indication.

$$A) \text{♩} = 80 \text{ MM} \quad | \quad \text{♩} = 100 \text{ MM}$$

$$B) \text{♩} = \overset{3}{\text{♩}}$$

5b) Calculating metric modulations.

In A) above, the performer may want to know the relation between the beat of the first tempo to that of the second. In B), the performer may want to know what the metronome indication is after (or before) the change of tempo. (In this case, we have to know the metronome indication of the music either before or after the tempo change.)

Some times the composer gives both A) and B) at once; if so, he or she has to know how A) and B) are related. If we understand this, then we can calculate B) from A) and vice versa. We start by setting up two metronome tempo indications.

$$N_1 = B_1 \text{ MM} \quad \text{and} \quad N_2 = B_2 \text{ MM}$$

$$\text{and } N_1 \text{ (in seconds)} = N_2 \text{ (in seconds)}.$$

5c) Relating tempos (part 1).

We define a variable P that relates the first tempo to the second tempo.

$$\text{This gives: } PN_1 = P \frac{60}{B_1} = \frac{60}{B_2} = N_2$$

We can determine P from knowing both N_1 and N_2 , or B_1 and B_2 .

$$PN_1 = N_2 \text{ and } PB_2 = B_1 \text{ (derived from } P \frac{60}{B_1} = \frac{60}{B_2} \text{)}$$

5d) Relating tempos (part 2).

If P relates the first tempo to the second, $\frac{1}{P}$ relates the second tempo to the first

This is why:

$$\text{We have } PN_1 = P \frac{60}{B_1} = \frac{60}{B_2} = N_2$$

Multiplying everything times $\frac{1}{P}$ we get

$$\frac{1}{P}PN_1 = \frac{1}{P}P \frac{60}{B_1} = \frac{1}{P} \frac{60}{B_2} = \frac{1}{P}N_2$$

Simplifying,

$$N_1 = \frac{60}{B_1} = \frac{1}{P} \frac{60}{B_2} = \frac{1}{P}N_2 \text{ and}$$

$$N_1 = \frac{1}{P}N_2 \text{ and } B_2 = \frac{B_1}{P}$$

5e) Converting metric modulation A) into B).

For this example, we go back to section 5a) and work with the indication

$$\downarrow = 80 \text{ MM} \quad | \quad \downarrow = 100 \text{ MM}$$

Let us set N_1 to Q , and B_1 to 80. Thus in the first tempo $N_1 = Q = \frac{4}{3}$ seconds.

We set B_2 to 100. We can solve for P , using the formula, $P \frac{60}{B_1} = \frac{60}{B_2}$

$$P \frac{60}{80} = \frac{60}{100} = \frac{80}{100} = \frac{4}{5}$$

$$\text{So } P = \frac{4}{5}.$$

$$\text{Now } PN_1 = \frac{4}{5}N_1 = N_2. \text{ Since } N_1 = Q \text{ we let } N_1 = \frac{5Q}{5} \text{ and } PN_1 = \frac{4}{5} \times \frac{5Q}{5} = \frac{4Q}{5} = N_2$$

Thus, the duration of N_2 in the first tempo equals 4 quintuplet notes or $\overset{5}{\bullet}$

This means we can write the relation between the two tempi in the manner of B). We set N_2 to Q so we write:

$$\overset{5}{\bullet} = \bullet$$

Putting A) and B) together we have

$$\downarrow = 80 \quad (\overset{5}{\bullet} = \bullet) \quad \downarrow = 100.$$

There is a practical problem here if the music in the first tempo of ♩ = 80 does not have quintuple divisions. We can fix this by setting a new duration, M_1 in the first tempo to a new duration M_2 in the second tempo. The relation from the first tempo to the second is P , so $PM_1 = M_2$. (Remember that $P = \frac{5}{4}$.)

Now let $M_1 = Q$, $M_2 = PQ = \frac{5}{4}Q = Q + S$

so we write: ♩ = ♩ and

♩ = 80 MM (♩ = ♩) ♩ = 100 MM

5f) Converting metric modulation B) into A).

Here we will work with the equivalence supposedly found in a score:

♩ = ♩

The first tempo is defined as ♩ = 80 MM. We want to determine the second tempo,

that is, ♩ = ? MM.

Starting with the first tempo and the formula $d = \frac{60}{B}$ seconds, we have $Q = \frac{60}{80}$ seconds.

So $Q = \frac{3}{4}$ seconds. We have to find the value of ♩ or $5E$. From Q , $E = \frac{3}{8}$, and $5E = \frac{15}{8}$

or $1\frac{7}{8}$ seconds. Now we know that in the second tempo, ♩ = $1\frac{7}{8}$ seconds. Since the

tempo is to be set for quarter notes not half notes, $Q = \frac{H}{2} = \frac{15}{8} \times \frac{1}{2} = \frac{15}{16}$. Using the formula again, we have

$Q = \frac{60}{B}$ and $\frac{15}{16} = \frac{60}{B}$. $B = \frac{16 \times 60}{15} = 16 \times 4 = 64$. So the second tempo is ♩ = 64 MM.

So we can write:

$$\bullet = 80 \text{ MM} \quad (\text{♩} = \text{♩}) \quad \bullet = 64 \text{ MM}.$$

By the way, the value of P in $PN_1 = N_2$ is $\frac{4}{5}$ since $P(5E) = 4E$ and $P = \frac{4E}{5E} = \frac{4}{5}$.

6) Summary.

Given “B MM”...

$$d = \frac{60}{B} \text{ seconds: \# of seconds per duration between clicks.}$$

$$\frac{1}{d} = \frac{B}{60} \text{ seconds: \# of durations per second.}$$

Tempos are given by the marking “N = B MM”, where N is a rhythmic value and B is a number.

In metric modulations with two tempi, respectively $N_1 = B_1 \text{ MM}$ and $N_2 = B_2 \text{ MM}$...

$$PN_1 = P \frac{60}{B_1} = \frac{60}{B_2} = N_2.$$

P relates the first tempo to the second, such that

$$PN_1 = N_2 \quad \text{and} \quad PB_2 = B_1$$

and

$$N_1 = \frac{1}{P} N_2 \quad \text{and} \quad B_2 = \frac{B_1}{P}.$$