Constructing Invariance Matrices and Vectors for Two Pcsets.[1]

Robert Morris (2016)

Given two (unordered) pcsets, A and B, we wish to determine #(A ∩ T_nB) and #(A ∩ T_nIB).[2] There is no limitation on the content or cardinality of A or B.

To find out we construct two matrices from A and B. The first is called a **T-matrix** and the second called an **I-matrix**. From these we construct **T-vectors** and **I-vectors**, respectively.

Constructing the T-matrix

To make the T-matrix we construct a two-dimensional table from A and the inversion of B.

We set up the T-matrix with A as its head row and IB as its head column. To illustrate this we define A = {035} and B = {458}. (Note: IB = {874})

	0	3	5
8			
7			
4			

We fill in the matrix with the mod-12 sums of the row and column heads[3] as follows.

	0	3	5
8	8	B	1
7	7	A	0
4	4	7	9

From the contents of the T-matrix, we construct a T-vector. The T-vector starts out as a twelve-position array filled in with zeros, like so:

T-vector: [000000000000].

For each occurrence of the number n in the T-matrix we add 1 to the nth place in the T-vector.

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[1] Note that the names of the two pcsets A and B should not be confused with the names of the pcs A (= 10) and B (= 11).

[2] That is, how many pcs are in common between pcset A and each of the 12 transpositions of pcset B and how many pcs are in common between pcset A and each of the 12 transposed inversions of pcset B.

[3] Formally, for T-matrix M, $M_{i,j} = A_i + IB_j$ (mod 12). And for I-matrix M, $M_{i,j} = A_i + B_j$ (mod 12).

There is one 0 in the T-matrix, so we add 1 to the 0^{th} place in the T-vector.

[100000000000]

There is one 1 in the T-matrix, so we add 1 to the 1^{th} place in the T-vector.

[110000000000]

There is one instance of the numbers 4, 8, 9, A, and B, in the T-matrix, so we add 1 to the 4^{th}, 8^{th}, 9^{th}, A^{th}, and B^{th} places in the T-vector.

[110010001111]

There are two 7s in the T-matrix, so we add 2 to the 7^{th} place of the T-vector.

[110010021111]

The T-matrix is complete and the sum of its content (in this case 9) is the number of entries in the I-matrix (which is equal to the #A X #B).

We can read off $\#(A \cap T_nB)$ from the T-vector. The number in the n^{th} place in the T-vector is $\#(A \cap T_nB)$.

Thus: for T-vector [110010021101] A = {035} and B = {458}:

$1 = \#(A \cap T_0B)$ since the 0^{th} place in the T-vector is 1.
A = {035} and T_0B = {458}: one pc in common, pc 5.

$1 = \#(A \cap T_1B)$ since the 1^{th} place in the T-vector is 1.
A = {035} and T_1B = {569}: one pc in common, pc 5.

$0 = \#(A \cap T_2B)$ since the 2^{th} place in the T-vector is 0.
A = {035} and T_2B = {67A}: no pc in common.

$0 = \#(A \cap T_3B)$ since the 3^{rd} place in the T-vector is 0.
A = {035} and T_3B = {78B}: no pc in common.

. . .

$2 = \#(A \cap T_7B)$ since the 7^{th} place in the T-vector is 2.
A = {035} and T_7B = {B03}: two pcs in common, pcs 0 and 3.

Etc.

Constructing the I-matrix

To make the T-matrix we construct a two-dimensional table from A and B.

We set up the I-matrix with A as its head row and B as its head column. To illustrate this use the A and B pcsets from above. A = {035} and B = {458}.

	0	3	5
4			
5			
8			

We fill in the matrix with the mod-12 sums of the row and column heads as follows.

	0	3	5
4	4	7	9
5	5	8	A
8	8	B	1

From the contents of the I-matrix, we construct an I-vector. The I-vector starts out as a twelve-position array filled in with zeros:

I-vector: [000000000000].

For each occurrence of the number n in the I-matrix we add 1 to the nth place in the I-vector.

There is one 1 in the I-matrix, so we add 1 to the 1^{th} place in the I-vector, like so.

[010000000000]

There is one 4 in the I-matrix, so we add 1 to the 4^{th} place in the I-vector.

[010010000000]

There is one instance of the numbers 5, 7, 9, A, and B, in the I-matrix, so we add 1 to the 5^{th}, 7^{th}, 9^{th}, A^{th}, and $B^{th.}$ places in the I-vector.

[010011010111]

There are two 8s in the I-matrix, so we add 2 to the 8^{th} place of the I-vector.

[010011012111]

The I-matrix is complete and the sum of its content (in this case 9) is the number of entries in the I-matrix (which is equal to the #A X #B).

We can read off #(A ∩ T_nIB) from the I-vector. The number in the n^{th} place in the I-vector is #(A ∩ T_nIB).

Thus: for T-vector [010011012111] A = {035} and B = {458} (and IB = {874}:

0 = #(A ∩ T_0IB) since the 0^{th} place in the I-vector is 0.
A = {035} and IT_0B = {874}: no pcs in common.

1 = #(A ∩ T_1IB) since the 1^{th} place in the I-vector is 1.
A = {035} and IT_1B = {985}: one pc in common, pc 5.

0 = #(A ∩ T_2IB) since the 2^{th} place in the I-vector is 0.
A = {035} and IT_2B = {A96}: no pcs in common.

0 = #(A ∩ T_3IB) since the 3^{rd} place in the I-vector is 0.
A = {035} and IT_3B = {BA7}: no pcs in common.

1 = #(A ∩ T_4IB) since the 4^{th} place in the I-vector is 1.
A = {035} and IB = {0B8}: 1 pc in common, pc 0.

. . .

2 = #(A ∩ T_8IB) since the 8^{th} place in the I-vector is 2.
A = {035} and IB = {430}: two pcs in common, pcs 0 and 4.

Etc.

T- and I-matrices for other pairs of pcsets.

Example 1.

A = {1258}
B = {368}

T-matrix A = column head; IB = row head.

	1	2	5	8
9	A	B	2	5
6	7	8	B	2
4	5	6	9	0

T-vector (The sum of its content is 12)

[102002111112]

1 = #(A ∩ T_0B) since the 0^{th} place in the T-vector is 1.
A = {1258} and B = {368}: one pc in common, pc 8.

2 = #(A ∩ T_2B) since the 2nd place in the T-vector is 2.
A = {1258} and T_2B = {589}: two pcs in common, pcs 5 and 8.

2 = #(A ∩ T_BB) since the Bth place in the T-vector is 2.
A = {1258} and T_BB = {257}: two pcs in common, pcs 2 and 5.

Etc.

I-matrix A = column head; B = row head.

	1	2	5	8
3	4	5	8	B
6	7	8	B	2
8	9	A	1	4

I-vector

[011021012112]

0 = #(A ∩ T_0IB) since the 0th place in the I-vector is 0.
A = {1258} and IB = {964}: no pcs in common.

2 = #(A ∩ T_4IB) since the 4th place in the I-vector is 2.
A = {1258} and T_4IB = {1A8}: two pcs in common, pcs 1 and 8.

2 = #(A ∩ T_BIB) since the Bth place in the I-vector is 2.
A = {1258} and T_BIB = {853}: two pcs in common, pcs 5 and 8.

Etc.

Example 2.

A = {0236AB}
B = {0236AB}

Here the two pcsets are the same, but we will still call them A and B.[4]

T-matrix A = column head; IB = {0A9621} = row head.

	0	2	3	6	A	B
0	0	2	3	6	A	B
A	A	0	1	4	8	9
9	9	B	0	3	7	8
6	6	8	9	0	4	5
2	2	4	5	8	0	1
1	1	3	4	7	B	0

T-vector (content sum is 36.)

[632342224323]

From the T-vector, we see that every transposition of B (=A) has common pcs with A. (There are n zeros in the T-vector.) Aside from the intersection of A and B (i.e., A with itself), the maximum cardinality of intersection is 4 between A and T_4B and A and T_8B:

4 = #(A ∩ T_4B) since the 4th place in the T-vector is 4.
A = {0236AB} and T_4B = {467A23}: four pcs in common: 6, A, 2, 3.

4 = #(A ∩ T_8B) since the 8th place in the T-vector is 4.
A = {0236AB} and T_8B = {8AB267}: four pcs in common: A, B, 2, 6.

Etc.

[4] We could use the ICV of pcset A to determine the cardinality of intersection between A and transpositions of itself. The ICV(A) = [6323421]. Note the correspondence between the T-vector and the IVC up to the 5th place, while the 6th place in the ICV is the 6th place of the T-vector divided by 2. This holds for the comparison of ICV(A) and T-vectors for any pcset A and itself.

I-matrix A = column head; B = row head.

	0	2	3	6	A	B
0	0	2	3	6	A	B
2	2	4	5	8	0	1
3	3	5	6	9	1	2
6	6	8	9	0	4	5
A	A	0	1	4	8	9
B	B	1	2	5	9	A

I-vector

[444234303432]

The minimal cardinality of intersection is 0; A and T_7IB have no pcs in common: A = {0236AB} and T_7IB = {754198}. The 7[th] place of the I-vector is 0.

Maximal intersection of pcs is 4 with T_0IB, T_1IB, T_2IB, T_5IB, and T_9IB. The I-vector has 4 in its places 0, 1, 2, 5, and 9. This is shown with the intersecting pcs underlined below.

A = $\{\underline{02}3\underline{6A}B\}$ $T_0IB = \{\underline{0A}9\underline{62}1\}$

A = $\{0\underline{23}6\underline{AB}\}$ $T_1IB = \{1\underline{BA}7\underline{32}\}$

A = $\{\underline{023}6A\underline{B}\}$ $T_2IB = \{\underline{20B}84\underline{3}\}$

A = $\{0\underline{236}A\underline{B}\}$ $T_5IB = \{5\underline{32B}7\underline{6}\}$

A = $\{02\underline{36AB}\}$ $T_9IB = \{97\underline{63BA}\}$

Example 3

A = {023458}
B = {023467}

T-matrix A = column head; IB = {0A9865} = row head.

	0	2	3	4	5	8
0	0	2	3	4	5	8
A	A	0	1	2	3	6
9	9	B	0	1	2	5
8	8	A	B	0	1	4
6	6	8	9	A	B	2
5	5	7	8	9	A	1

T-vector (content sum is 36)

[444223214343]

The minimal cardinality of intersection—1 pc—is between A and T_7B. The 7th place of the T-vector is 1. A = {023458}; T_7B = {79AB12}. The shared pc is 2.

The maximum number of shared pcs is 4 under five distinct transpositions of B.

A = {023458} and T_0B = {023467} T-matrix place 0 = 4.

A = {023458} and T_1B = {124578} T-matrix place 1 = 4.

A = {023458} and T_2B = {235689} T-matrix place 2 = 4.

A = {023458} and T_8B = {8AB023} T-matrix place 8 = 4.

A = {023458} and T_AB = {A01234} T-matrix place A = 4.

I-matrix A = column head; B = row head.

	0	2	3	4	5	8
0	0	2	3	4	5	8
2	2	4	5	6	7	A
3	3	5	6	7	8	B
4	4	6	7	8	9	0
6	6	8	9	A	B	2
7	7	9	A	B	0	3

I-vector

[303333444333]

A and T_1IB are disjoint (complements) since I-vector place 1 = 0.

A = {023458} and T_1IB = {1BA976}

3 = #(A ∩ T_nIB), for n = 0, 2, 3, 4, 5, 9, A, B since I-vector place n = 3

For example, A = {023458} and T_0IB = {0A9865}; three pcs are in common.

4 = #(A ∩ T_nIB), for n = 6, 7, 8 since I-vector place n = 4.

For example, A = {023458} and T_6IB = {64320B}; four pcs are in common.

So, in general, for A = {023458} and B = {023467}, the cardinality of intersection between A and T_nB and/or T_nIB is 0 or 1 or 2 or 3 or 4.

Optional Homework

Make T and I-matrices and vectors for the pair A = {013458} and B = {123789}.

Make T- and I-matrices and vectors for A = B = the whole-tone scale.

Make T- and I-matrices and vectors for A = {01356} and B = (the complement of A).