Constructing Invariance Matrices and Vectors for Two Pcssets.\(^1\)

Robert Morris (2016)

Given two (unordered) pcssets, A and B, we wish to determine \(\#(A \cap T_n B)\) and \(\#(A \cap T_n I B)\).\(^2\) There is no limitation on the content or cardinality of A or B.

To find out we construct two matrices from A and B. The first is called a **T-matrix** and the second called an **I-matrix**. From these we construct **T-vectors** and **I-vectors**, respectively.

### Constructing the T-matrix

To make the T-matrix we construct a two-dimensional table from A and the inversion of B.

We set up the T-matrix with A as its head row and IB as its head column. To illustrate this we define A = \{035\} and B = \{458\}. (Note: IB = \{874\})

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>3</th>
<th>5</th>
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</thead>
<tbody>
<tr>
<td>8</td>
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<td>7</td>
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<td>4</td>
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</tbody>
</table>

We fill in the matrix with the mod-12 sums of the row and column heads\(^3\) as follows.

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<tr>
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<tbody>
<tr>
<td>8</td>
<td>8</td>
<td>B</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>A</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>7</td>
<td>9</td>
</tr>
</tbody>
</table>

From the contents of the T-matrix, we construct a T-vector. The T-vector starts out as a twelve-position array filled in with zeros, like so:

T-vector: \[000000000000\].

For each occurrence of the number \(n\) in the T-matrix we add 1 to the \(n\)th place in the T-vector.

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\(^1\) Note that the names of the two pcssets A and B should not be confused with the names of the pcs A (=10) and B (=11).

\(^2\) That is, how many pcs are in common between pcsset A and each of the 12 transpositions of pcsset B and how many pcs are in common between pcsset A and each of the 12 transposed inversions of pcsset B.

\(^3\) Formally, for T-matrix \(M\), \(M_{ij} = A_i + IB_i \pmod{12}\). And for I-matrix \(M\), \(M_{ij} = A_i + B_j \pmod{12}\).
There is one 0 in the T-matrix, so we add 1 to the 0\(^{th}\) place in the T-vector.

\[100000000000\]

There is one 1 in the T-matrix, so we add 1 to the 1\(^{st}\) place in the T-vector.

\[110000000000\]

There is one instance of the numbers 4, 8, 9, A, and B, in the T-matrix, so we add 1 to the 4\(^{th}\), 8\(^{th}\), 9\(^{th}\), A\(^{th}\), and B\(^{th}\) places in the T-vector.

\[110010011111\]

There are two 7s in the T-matrix, so we add 2 to the 7\(^{th}\) place of the T-vector.

\[110010021111\]

The T-matrix is complete and the sum of its content (in this case 9) is the number of entries in the I-matrix (which is equal to the \(\#A \times \#B\)).

We can read off \(\#(A \cap T_n B)\) from the T-vector. The number in the \(n^{th}\) place in the T-vector is \(\#(A \cap T_n B)\).

Thus: for T-vector [110010021111] \(A = \{035\}\) and \(B = \{458\}\):

1 \(= \#(A \cap T_0 B)\) since the 0\(^{th}\) place in the T-vector is 1.

\(A = \{035\}\) and \(T_0 B = \{458\}\): one pc in common, pc 5.

1 \(= \#(A \cap T_1 B)\) since the 1\(^{st}\) place in the T-vector is 1.

\(A = \{035\}\) and \(T_1 B = \{569\}\): one pc in common, pc 5.

0 \(= \#(A \cap T_2 B)\) since the 2\(^{nd}\) place in the T-vector is 0.

\(A = \{035\}\) and \(T_2 B = \{67A\}\): no pc in common.

0 \(= \#(A \cap T_3 B)\) since the 3\(^{rd}\) place in the T-vector is 0.

\(A = \{035\}\) and \(T_3 B = \{78B\}\): no pc in common.

\[ \ldots \]

2 \(= \#(A \cap T_7 B)\) since the 7\(^{th}\) place in the T-vector is 2.

\(A = \{035\}\) and \(T_7 B = \{B03\}\): two pcs in common, pcs 0 and 3.

Etc.
Constructing the I-matrix

To make the T-matrix we construct a two-dimensional table from A and B.

We set up the I-matrix with A as its head row and B as its head column. To illustrate this use the A and B pcsets from above. \( A = \{035\} \) and \( B = \{458\} \).

\[
\begin{array}{c|ccc}
& 0 & 3 & 5 \\
4 &   &   & \\
5 &   &   & \\
8 &   &   & \\
\end{array}
\]

We fill in the matrix with the mod-12 sums of the row and column heads as follows.

\[
\begin{array}{c|ccc}
& 0 & 3 & 5 \\
4 & 4 & 7 & 9 \\
5 & 5 & 8 & A \\
8 & 8 & B & 1 \\
\end{array}
\]

From the contents of the I-matrix, we construct an I-vector. The I-vector starts out as a twelve-position array filled in with zeros:

I-vector: \([000000000000]\).

For each occurrence of the number \( n \) in the I-matrix we add 1 to the \( n \)th place in the I-vector.

There is one 1 in the I-matrix, so we add 1 to the 1\(^{st}\) place in the I-vector, like so.

\([010000000000]\)

There is one 4 in the I-matrix, so we add 1 to the 4\(^{th}\) place in the I-vector.

\([010010000000]\)

There is one instance of the numbers 5, 7, 9, A, and B, in the I-matrix, so we add 1 to the 5\(^{th}\), 7\(^{th}\), 9\(^{th}\), A\(^{th}\), and B\(^{th}\) places in the I-vector.

\([010011010111]\)

There are two 8s in the I-matrix, so we add 2 to the 8\(^{th}\) place of the I-vector.

\([010011012111]\)

The I-matrix is complete and the sum of its content (in this case 9) is the number of entries in the I-matrix (which is equal to the \( \#A \times \#B \)).
Invariance Matrices and Vectors

We can read off \( #(A \cap T_nIB) \) from the I-vector. The number in the \( n \)th place in the I-vector is \( #(A \cap T_nIB) \).

Thus: for T-vector \([010011012111]\) \( A = \{035\} \) and \( B = \{458\} \) (and \( IB = \{874\} \):

\[
0 = #(A \cap T_0IB) \text{ since the 0th place in the I-vector is 0.} \\
A = \{035\} \text{ and } IT_0B = \{874\}: \text{ no pcs in common.}
\]

\[
1 = #(A \cap T_1IB) \text{ since the 1th place in the I-vector is 1.} \\
A = \{035\} \text{ and } IT_1B = \{985\}: \text{ one pc in common, pc 5.}
\]

\[
0 = #(A \cap T_2IB) \text{ since the 2th place in the I-vector is 0.} \\
A = \{035\} \text{ and } IT_2B = \{A96\}: \text{ no pcs in common.}
\]

\[
0 = #(A \cap T_3IB) \text{ since the 3rd place in the I-vector is 0.} \\
A = \{035\} \text{ and } IT_3B = \{BA7\}: \text{ no pcs in common.}
\]

\[
1 = #(A \cap T_4IB) \text{ since the 4th place in the I-vector is 1.} \\
A = \{035\} \text{ and } IB = \{0B8\}: \text{ 1 pc in common, pc 0.}
\]

\[
\ldots
\]

\[
2 = #(A \cap T_8IB) \text{ since the 8th place in the I-vector is 2.} \\
A = \{035\} \text{ and } IB = \{430\}: \text{ two pcs in common, pcs 0 and 4.}
\]

Etc.

T- and I-matrices for other pairs of pcsets.

Example 1.

\[
A = \{1258\} \\
B = \{368\}
\]

T-matrix \( A = \) column head; \( IB = \) row head.

\[
\begin{array}{cccc}
1 & 2 & 5 & 8 \\
9 & A & B & 2 & 5 \\
6 & 7 & 8 & B & 2 \\
4 & 5 & 6 & 9 & 0
\end{array}
\]

T-vector (The sum of its content is 12)

\[
[102002111112]
\]

1 = \( #(A \cap T_0B) \) since the 0th place in the T-vector is 1.
\( A = \{1258\} \) and \( B = \{368\} \): one pc in common, pc 8.
2 = #(A ∩ T₂B) since the 2nd place in the T-vector is 2.
   A = \{1258\} and T₂B = \{589\}: two pcs in common, pcs 5 and 8.

2 = #(A ∩ T₅B) since the 5th place in the T-vector is 2.
   A = \{1258\} and T₅B = \{257\}: two pcs in common, pcs 2 and 5.

Etc.

I-matrix A = column head; B = row head.

<table>
<thead>
<tr>
<th></th>
<th>1 2 5 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4 5 8 B</td>
</tr>
<tr>
<td>6</td>
<td>7 8 B 2</td>
</tr>
<tr>
<td>8</td>
<td>9 A 1 4</td>
</tr>
</tbody>
</table>

I-vector

[011021012112]

0 = #(A ∩ T₀B) since the 0th place in the I-vector is 0.
   A = \{1258\} and T₀B = \{964\}: no pcs in common.

2 = #(A ∩ T₄B) since the 4th place in the I-vector is 2.
   A = \{1258\} and T₄B = \{1A8\}: two pcs in common, pcs 1 and 8.

2 = #(A ∩ T₈B) since the 8th place in the I-vector is 2.
   A = \{1258\} and T₈B = \{853\}: two pcs in common, pcs 5 and 8.

Etc.
Example 2.

\[ \begin{align*}
A &= \{0236AB\} \\
B &= \{0236AB\}
\end{align*} \]

Here the two pcsets are the same, but we will still call them A and B.\(^4\)

T-matrix \(A\) = column head; \(IB = \{0A9621\} = \) row head.

\[
\begin{array}{c|cccccc}
& 0 & 2 & 3 & 6 & A & B \\
\hline
0 & 0 & 2 & 3 & 6 & A & B \\
A & A & 0 & 1 & 4 & 8 & 9 \\
9 & 9 & B & 0 & 3 & 7 & 8 \\
6 & 6 & 8 & 9 & 0 & 4 & 5 \\
2 & 2 & 4 & 5 & 8 & 0 & 1 \\
1 & 1 & 3 & 4 & 7 & B & 0 \\
\end{array}
\]

T-vector (content sum is 36.)

\[ [632342224323] \]

From the T-vector, we see that every transposition of B (=A) has common pcs with A. (There are \(n\) zeros in the T-vector.) Aside from the intersection of A and B (i.e., A with itself), the maximum cardinality of intersection is 4 between A and \(T_4B\) and A and \(T_8B\):

\[
4 = \#(A \cap T_4B) \text{ since the } 4^{th} \text{ place in the T-vector is } 4. \\
\quad A = \{0236AB\} \text{ and } T_4B = \{467A23\}: \text{ four pcs in common: } 6, A, 2, 3.
\]

\[
4 = \#(A \cap T_8B) \text{ since the } 8^{th} \text{ place in the T-vector is } 4. \\
\quad A = \{0236AB\} \text{ and } T_8B = \{8AB267\}: \text{ four pcs in common: } A, B, 2, 6.
\]

Etc.

\[ \]

\(^4\) We could use the ICV of pcset A to determine the cardinality of intersection between A and transpositions of itself. The ICV(A) = [6323421]. Note the correspondence between the T-vector and the IVC up to the 5\(^{th}\) place, while the 6\(^{th}\) place in the ICV is the 6\(^{th}\) place of the T-vector divided by 2. This holds for the comparison of ICV(A) and T-vectors for any pcset A and itself.
I-matrix $A = \text{column head}; B = \text{row head}.$

\[
\begin{array}{c|cccccc}
 & 0 & 2 & 3 & 6 & A & B \\
0 & 0 & 2 & 3 & 6 & A & B \\
2 & 2 & 4 & 5 & 8 & 0 & 1 \\
3 & 3 & 5 & 6 & 9 & 1 & 2 \\
6 & 6 & 8 & 9 & 0 & 4 & 5 \\
A & A & 0 & 1 & 4 & 8 & 9 \\
B & B & 1 & 2 & 5 & 9 & A \\
\end{array}
\]

I-vector

\[444234303432\]

The minimal cardinality of intersection is 0; $A$ and $T_{7IB}$ have no pcs in common: $A = \{0236AB\}$ and $T_{7IB} = \{754198\}$. The 7th place of the I-vector is 0.

Maximal intersection of pcs is 4 with $T_{0IB}$, $T_{1IB}$, $T_{2IB}$, $T_{5IB}$, and $T_{9IB}$. The I-vector has 4 in its places 0, 1, 2, 5, and 9. This is shown with the intersecting pcs underlined below.

$A = \{0236AB\}$ $T_{0IB} = \{0A9621\}$

$A = \{0236AB\}$ $T_{1IB} = \{1BA732\}$

$A = \{0236AB\}$ $T_{2IB} = \{20B843\}$

$A = \{0236AB\}$ $T_{5IB} = \{532B76\}$

$A = \{0236AB\}$ $T_{9IB} = \{9763BA\}$
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Example 3

A = \{023458\}
B = \{023467\}

T-matrix A = column head; IB = \{0A9865\} = row head.

\[
\begin{array}{cccccc}
0 & 2 & 3 & 4 & 5 & 8 \\
0 & 2 & 3 & 4 & 5 & 8 \\
A & A & 0 & 1 & 2 & 3 \\
9 & 9 & B & 0 & 1 & 2 \\
8 & 8 & A & B & 0 & 1 \\
6 & 6 & 8 & 9 & A & B \\
5 & 5 & 7 & 8 & 9 & A
\end{array}
\]

T-vector (content sum is 36)

[444223214343]

The minimal cardinality of intersection—1 pc—is between A and T_B. The 7th place of the T-vector is 1. A = \{023458\}; T_B = \{79AB12\}. The shared pc is 2.

The maximum number of shared pcs is 4 under five distinct transpositions of B.

A = \{023458\} and T_0B = \{023467\} T-matrix place 0 = 4.
A = \{023458\} and T_1B = \{124578\} T-matrix place 1 = 4.
A = \{023458\} and T_2B = \{235689\} T-matrix place 2 = 4.
A = \{023458\} and T_3B = \{8AB023\} T-matrix place 3 = 4.
A = \{023458\} and T_4B = \{A01234\} T-matrix place A = 4.

I-matrix A = column head; B = row head.

\[
\begin{array}{cccccc}
0 & 2 & 3 & 4 & 5 & 8 \\
0 & 2 & 3 & 4 & 5 & 8 \\
2 & 2 & 4 & 5 & 6 & 7 \\
3 & 3 & 5 & 6 & 7 & 8 \\
4 & 4 & 6 & 7 & 8 & 9 \\
6 & 6 & 8 & 9 & A & B \\
7 & 7 & 9 & A & B & 0 \\
\end{array}
\]

I-vector

[303333444333]

A and T_1IB are disjoint (complements) since I-vector place 1 = 0.
A = \{023458\} and T_{1IB} = \{1BA976\}

3 = \#(A \cap T_nIB), for n = 0, 2, 3, 4, 5, 9, A, B since I-vector place n = 3

For example, A = \{023458\} and T_{0IB} = \{0A9865\}; three pcs are in common.

4 = \#(A \cap T_nIB), for n = 6, 7, 8 since I-vector place n = 4.

For example, A = \{023458\} and T_{6IB} = \{64320B\}; four pcs are in common.

So, in general, for A = \{023458\} and B = \{023467\}, the cardinality of intersection between A and T_nB and/or T_nIB is 0 or 1 or 2 or 3 or 4.

**Optional Homework**

Make T and I-matrices and vectors for the pair A = \{013458\} and B = \{123789\}.

Make T- and I-matrices and vectors for A = B = the whole-tone scale.

Make T- and I-matrices and vectors for A = \{01356\} and B = (the complement of A).