Bob's Atonal Theory Primer[1]

Pitch and pitch-class (pc)

(1) *Pitch space*: a linear series of pitches (semitones) from low to high modeled by integers.
(2) Sets of pitches (called *psets*) are selections from the set of pitches; they are unordered in time.

(3) *Pc space*: circle of pitch-classes (no lower or higher relations) modeled by integers, mod 12 (see below).
(4) Pcs are related to the pitches by taking the latter mod 12. Pitches related by any number of octaves map to the same pitch-class.

(5) Sets of pcs (called *pcsets*) are selections from the set of pcs; they are unordered in time (and pitch).

(6) Pcsets must be "realized" (or "represented" or "articulated") by pitches. To realize a pcset in music, it must be ordered in pitch and in time. Every musical articulation of a pcset produces a contour. Many different psets may represent one pcset. Pcsets may model melodies, harmonies, mixed textures, etc.

Definitions from Finite Set Theory

(6) The set of all the pcs is called the *aggregate* and is denoted by the letter U; the set of no pcs is called the empty or *null* set, and is denoted by the sign $\varnothing$

(7) Membership: If a is a member (or element) of the set B, we write $a \in B$.
(8) Inclusion: If A and B are sets and A is contained in B, we write $A \subset B$.
(9) The union of two sets A and B (written $A \cup B$) is the content of both of them.
(10) The intersection of two sets A and B is their common elements (written $A \cap B$).
(11) Two sets are disjoint if their intersection is $\varnothing$.
(12) B is the complement of A, if B contains all elements of U not in A. We show the complement of A by A'.
NB: $A \cap A' = \varnothing$ (A and A' are disjoint) and $A \cup A' = U$.

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[1] We do not use the usual (but inappropriate) name "Set Theory" for this subject.

Definitions and relations

	entity	ordered interval	unordered interval	series; written within < and >	collections; written within { and }	transposition
pitch	0 = middle C = C_4; -24 cello C = C_2; 6 = F#$_4$; etc.	from a to b = (b-a)	"distance" between pitches a and b = $\lvert b-a \rvert$ = $\lvert a-b \rvert$	**pseg** Example <7,7,7,3>	**pset** Example {-3, 1,4}	Let pitch b = the pitch transposition of pitch a by n semitones: We write b = $T_n a$ .and b = a + n. Example: 3 = $T_7(-4)$
pc	0 = C, 1 = C#,Db, B##; 2 = D, 3 = Eb, D#, etc., 10 = A#, Bb; 11 = B, Cb.	from a to b = (b-a) take mod-12; varies from 0 to 11;	"distance" between pcs a and b = $\lvert b-a \rvert$ = $\lvert a-b \rvert$ taken mod 12; varies from 0 to 6. Pc unordered intervals are called **interval classes** (**ic**s).	**pcseg**; Example <0,3,11,2> (NB: rows = ordering of all 12 pcs without duplication)	**pcset**, or **pc collection** Example {0157}	Let pc b = the pc transposition of pc a by n.: We write b = $T_n a$ .and b = a + n, mod 12. Example 3 = $T_7(8)$

remarks and explanation:

The pc 10 is written as "A" (or" a" or "T" or "t")

The pc 11 is written as "B" (or "b" or "E" or "e").

mod 12: If a number n is outside the range of 0 to 11, reduce or augment n by 12s until it falls within that range. –3 = 9 mod 12; 35 = B mod 12. Mod 12 models octave equivalence.

$|x|$ = is called the absolute value of x. $|x| = |-x|$. (Absolute value models distance, because the distance from a to b = the distance from b to a.)

T_0 has no effect on a pitch, pc, pset, or pcset; T_0 is the called the *identity operator*.

The elements in a pset or pcset can be written in any order, so {024} = {204} = {402}. etc. We just usually put them in ascending order for easy reading.

	inversion	**transposition after inversion**	**invariance**	**types of collections** (sets of collections)
pitch	Let pitch b = pitch inversion of pitch a: We write b = Ia and b = -a Example 4 = I-4 -22 = I22	Let pitch b = the transposition of the inversion of pitch a by n semitones; We write T_nIa and b= n-a Example -1 = T_6I7	if pset X = T_nIX, we say X is invariant. (NB: no pset is invariant under T_n, where n is not 0).) Example Let X = {-3,1,5}; X is invariant under T_2I.	**pset-class** is a set of all psets related by pitch space transposition and/or inversion
pc	Let pc b = pc inversion of pitch a: We write b = Ia .and b = -a, mod 12. Examples 4 = I8; 9 = I3.	Let pc b = the transposition of the inversion of pitch a by n semitones; We write T_nIa and b= n-a Example B = T_6I7	if pcset X = T_nX and/or T_nIX, we say X is invariant. Example X = {0167}; X is invariant under: T_6, T_1I and T_7I.	**Set-class** (abbr. **SC**) or **collection-class or set-type**; there are names for a set of all psets related by pitch-class transposition and/or inversion
remarks	This is inversion "around" 0 (pitch C_4 or pc C)	(1) If b = T_na then b = n-a and b+a = n n is called the inversional index. (2) If b = T_na then a and b are disposed "around" n/2. (3) There are two kinds of inversions, where the index n is even or odd. If even, then n/2 is a pitch or pc; if odd, then n/2 is inbetween two pitches or pcs.	(1) All sets are invariant under T_0 (2) The degree of invariance of a set is the number of operations, which produce invariance. All sets have at least 1 degree of invariance. Others have more; pset {-3 ,1,5} has a degree of invariance of 2; pcset {0167} has a degree of invariance of 4. (3) sets related by transposition and/or inversion have the same degree of invariance.	(1) Set-classes are represented by one of their members called a prime-form or normal order representative. (2) We can write set-classes as sets of pcsets: The set-class containing the major and minor chord can be written: { {037} {148}{259} {36A}, etc., {047} {158}{269}{37A} etc., {A25}{B36} } (3) The number of sets in a set-class = 24 divided by the degree of invariance of any of its members. Example. The set-class including {0167} has 6 members, since its degree of invariance is 4 and 24/4 = 6.

Interval-class vectors

The interval-classes in a pcset can be registered in a construct called an interval-class vector (IVC). An ICV consists of seven successive numbers within brackets. The leftmost number gives the number of interval-classes of size 0 (and thus, the cardinality of the SC's members), the second number from the left gives the number of interval-classes of size 1, and so forth until we get to the last (seventh, rightmost) number, which indicates the amount of ic6s in any set within the set-class. For example, [3011010] indicates that any set within the set-class with which it is associated has 3 pcs, no ic1s, one ic2, one ic3, no ic4 or 6, and one ic5. The ICV helps profile the "intervallic sound" of a set as well as determine the number of common tones shared by a pcset and any of its transpositions (see below)

All pcset members of a SC share the same ICV because an ic does not change under transposition and/or inversion.. The musical realizations of the members of a SC therefore have a similar sound.

Set and set-class relations

Relations among pcsets are called **literal**. They can be described and notated with the elementary concepts and notations given above in "Definitions from Finite Set Theory."

Relations among set-classes are called **abstract**.

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	inclusion	**complementation**	**Z-relation**
literal	For pcsets A and B: if B includes all pcs in A, then A ⊂ B. Example: A = {035}; B = {03459}	For pcsets A and B: If B includes all pcs not in A, then B = A′. Example A = {03479}; B = A′.= {12568AB}	For pcsets A and B: If and A and B are not both members of the same SC, and the ICV of A = the ICV of B, then A and B are Z-related. Example A = [1257}; B = {1349}; both have ICV[4111111]
abstract	SC X is included in SC Y, if, for pcset A ∈ X and pcset B ∈ Y, A ⊂ B.	SC X is the complement of SC Y, if, for A ∈ X and B ∈ Y, B = A′.	SC X is Z-related to SC Y, if, for A ∈ X and B ∈ Y, A is Z-related to B.
remarks		One of two complementary SCs, possesses the complements of the pcsets in the other. The two SCs have the same number after the hyphen in their names. Example SC 3-2[013] and SC 9-2[012345679A] are complementary SCs.	(1) Z-relations only relate pairs of pcsets or SCs. (2) Z-related pcsets have a similar intervallic sound, even though they are not related under transposition and/or inversion. (3) If one member of a pair of Z-related pcsets or SCs does not contain or is not contained in the complement of the other, the two are designated **ZC-related**.

Three basic theorems

1. *Transpositional Common Tone Theorem*: Let A and B be pcsets. The number of common pcs between A and T_nB is equal to the number of instances of interval n spanning from A to B.

   Corollary: The number of common pcs between A and T_nA is equal to the number of ics n in the ICV of A (except for ic 6, where the number of common pcs is twice the number of ic 6s).

2. *Inversional Common Tone Theorem*: The number of common pcs between A and T_nIB is equal to the number of instances of sums $a+b = n$, where $a \in A$ and $b \in B$.

   Corollary: The number of common pcs between A and T_nIA is equal to the number of instances of sums $a+b = n$, where a and $b \in A$. NB: instance $a + b$ is distinct from instance $b + a$, unless $a = b$.

3. *Complement Theorem*: For pcsets A and A′, the ICV of A′ is a transformation of the ICV of A: for each entry in the ICV of A except the entry for ic6, add k (for the entry of ic6, add k/2). $k = 2a - 12$, where a is the number of elements in A. (NB: $k = a - (12-a)$).

   Corollary (the Hexachord Theorem): If A is a hexachord, then $k = 0$, so complementary hexachords have the same ICV, as do their SCs.

   Remark. While complementary hexachords share the same ICV, they are not obliged to be members of the same SC; and if they are not, then they (and their SCS) are Z- and ZC-related.

—Robert Morris (2004)